Econ 714 Midterm #1 Solutions

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1. Farmer with market power

a. Bellman equation and conditions (20)

The farmer's Bellman equation is given by

$$\begin{aligned} V(s) &= \max_{q,s'} \left\{ p(q)q + \frac{1}{R} V(s') \right\} \qquad (1) \\ \text{subject to} \quad & 0 \le q \le s \\ & s \ge 0 \\ & s' = A(s-q) \end{aligned}$$

The first line says simply that the flow payoff is the current period's profits, and the future value is the value function evaluated at the farmer's choice of wheat stock tomorrow, discounted by the interest rate. The first constraint bounds the amount of the harvest between 0 (can't harvest negative) and the total stock (can't harvest what you don't have). The second constraint says that the wheat stock cannot go negative, and the final constraint is the law of motion for the wheat stock. Note that this last constraint is like a typical capital accumulation constraint.

Now that we know the Bellman equation, we can give the conditions for all of the remaining parts of the problem:

1. **Solvable:** The Bellman equation (1) is solvable when (i) the feasible correspondence defined by the constraint set, $\Gamma(s) = \{(q, s') | (q, s') \in [0, s] \times [0, As]\}$, is nonempty, compact-valued, and continuous, and (ii) the profit function $\pi(q) = p(q)q$ is bounded and continuous, with $0 < 1/R < 1$. Then the Bellman operator in (1) is a contraction and therefore has a unique fixed point V which corresponds to the value function.

2. **Increasing value function**: The value function V is strictly increasing when (i) for all s', $\pi(q) = \pi(s, s')$ is strictly increasing in s and (ii) Γ as defined above is monotone, so that $s \le \hat{s} \implies \Gamma(s) \subseteq \Gamma(\hat{s})$.

3. **Concave value function:** The value function V is strictly concave when (i) $\pi(q)$ is strictly concave and (ii) Γ is a convex correspondence.

b. Optimality conditions (25)

Now we can proceed to characterize the solution to (1) in the usual way. Throughout this problem, we assume that the conditions described in part a are satisfied. In addition, we assume that the function $\pi(q)$ is continuously differentiable over the relevant domain, so that under this assumption and the conditions discussed in part a the value function is continuously differentiable and we can take first order conditions.

We will characterize the solution, mindful that the constraint $q_t \leq s_t$ might bind (I assume some Inada condition around $q = 0$ to focus on only one direction of the inequality, but this case would be analogous). Given this, we can rewrite (1) in terms of only s' and s, considering only the dynamic variable in the usual way:

$$V(s) = \max_{s'} \left\{ p(s - s'/A)(s - s'/A) + \frac{1}{R} V(s') + \mu(s'/A) \right\}, \tag{2}$$

where μ is a Lagrange multiplier on the constraint that the harvest cannot exceed the stock. The FOC with respect to s' on (2) is

$$\frac{1}{A}[p'(q)q + p(q)] = \frac{1}{R} V'(s') + \frac{\mu}{A} \tag{3}$$

Note that throughout this problem, the prime superscript on p and V denote derivatives, and the prime on s and q denote next period values. Then, we can take an envelope condition and find that

$$V'(s) = p'(q)q + p(q). \tag{4}$$

We can plug (4) into (3) after updating it one period to find the optimality condition for the farmer (the Euler equation / inequality):

$$p'(q_t)q_t + p(q_t) \geq \frac{A}{R}[p'(q_{t+1})q_{t+1} + p(q_{t+1})] \tag{5}$$

Equation (5), together with the flow equation $s_{t+1} = A(s_t - q_t)$, determine the optimal harvesting policy of the farmer.

The next part of the question asks us to consider when the wheat production / harvesting q_t will be increasing over time. We can analyze this using the Euler inequality (5). Let's first consider the case when the constraint is not binding, so (5) holds with equality. The simplest way to solve this problem is to recognize that under our assumptions, the profit function $\pi(q) = p(q)q$ is concave, and so the Euler equation (5) becomes simply $\pi'(q) = \pi'(q')(A/R)$. Since π is concave, $q' > q \iff \pi'(q') < \pi'(q)$, or equivalently if $\pi'(q)/\pi'(q') = A/R > 1$.

Alternatively, let's define the elasticity of demand by $\epsilon_t = p_t/(q_t p'(q_t))$. Then, we can divide through by $p_t = p(q_t)$ in equation (5) to find that

$$\frac{1}{\epsilon_t} + 1 = \frac{A}{R}\left(\frac{1}{\epsilon_{t+1}} + 1\right)\frac{p_{t+1}}{p_t} \implies \frac{p_{t+1}}{p_t} = \frac{R\left(\frac{1}{\epsilon_t} + 1\right)}{A\left(\frac{1}{\epsilon_{t+1}} + 1\right)}. \tag{6}$$

Because p is decreasing in q by assumption, we know that $q_{t+1} > q_t$ if and only if $p_{t+1} < p_t$, or $p_{t+1}/p_t < 1$. This is the case when the numerator in (6) is larger than the denominator. Let's rewrite (6) in terms of markups:

$$\frac{p_{t+1}}{p_t} = \frac{R\left(\frac{1}{\epsilon_t} + 1\right)}{A\left(\frac{1}{\epsilon_{t+1}} + 1\right)} = \frac{R\mu_{t+1}}{A\mu_t}, \tag{7}$$

where $\mu_t \equiv \epsilon_t/(1+\epsilon_t)$ is the farmer's markup over marginal cost. Equation (7) tells us that the price is decreasing, and therefore the quantity is increasing, when the growth in markup μ_{t+1}/μ_t exceeds the discounted effective rate of return A/R. This result makes sense: the farmer will increase his quantity when the discounted future profit gain exceeds the current profit loss. Note that this second solution is actually in terms of endogenous objects, but I accepted it anyway.

In the case when the constraint is binding, (5) becomes a strict inequality and so $\pi'(q)/\pi'(q') > A/R$. In this case the condition on increasing quantity cannot be so easily pinned down, although $A/R > 1$ is still sufficient.

c. Equilibrium interest rate characterization (15)

In order to characterize the equilibrium interest rate in this model, we must first recognize that the Euler equation (5) determines the optimal supply in the market. In equilibrium, supply equals demand, and so we can just plug in demand in order to characterize the interest rate. To begin, let's plug in our functional forms and replace q with Q in (5):

$$-\epsilon Q_t^{-\epsilon-1}Q_t + Q_t^{-\epsilon} = \frac{A}{R}[-\epsilon Q_{t+1}^{-\epsilon-1}Q_{t+1} + Q_{t+1}^{-\epsilon}] \implies Q_t^{-\epsilon} = \frac{A}{R}Q_{t+1}^{-\epsilon}. \tag{8}$$

Then, plugging in the growth formula for aggregate demand for wheat, it follows immediately that the equilibrium interest rate is given by

$$R = A(1+g)^{-\epsilon}. \tag{9}$$

2. McCall model with capital accumulation

a. Bellman equations (25)

The Bellman equation for the employed worker is similar to the standard Bellman equation in a consumption-savings model given that there are no quits or separations:

$$\begin{aligned} W(w,a) &= \max_{c,a'} \left\{ u(c) + \beta W(w,a') \right\} \\ \text{subject to} \quad & a' = Ra + w - c \\ & c \geq 0, a \geq \bar{a} \end{aligned} \tag{10}$$

Let's recognize a few things about this equation. First, note that the first argument to the value function is w on both sides of the equals sign because the wage is constant. Additionally, employed workers face no uncertainty, so there is no expectation on the RHS of (10). Note that (10) makes it clear that the value of an employed worker is increasing in assets and the wage, since having more assets or a higher wage eases the budget constraint.

The Bellman equation for an unemployed worker is more interesting. Given the wording of the problem (and the original McCall model that we discussed in class), I assume the following timing: unemployed workers first receive their unemployment benefit z and make their consumption / saving decision, then receive a wage offer for a job they would start in the next period if they accept, and then finally they make their accept or reject decision. In the same way that this timing implied that the value of an unemployed worker was independent of a wage offer in class, now the only relevant state variable for an unemployed worker is his asset position a. From all this, it follows that

$$\begin{aligned} U(a) &= \max_{c,a'} \left\{ u(c) + \beta \int \max_{acc,rej} [U(a'), W(w,a')] dF(w) \right\} \\ \text{subject to} \quad & a' = Ra + z - c \\ & c \geq 0 \end{aligned} \tag{11}$$

Like (10), (11) is also increasing in a for the same reason. Together, equations (10) and (11) characterize the value of each type of worker in the economy.

b. Unemployed worker optimal strategy (15)

I will argue that the optimal strategy of an unemployed worker is a reservation wage, and that this reservation wage depends on their asset position. The same basic logic that led us to the reservation wage in the standard McCall model leads us to a reservation strategy here. To see this, fix the level of assets a. Taking the level of assets a as given, the consumption-savings and accept-reject decisions effectively separate given the monotonicity of each value function, U and W, in a. Thus, the optimal strategy will be a reservation wage.

The reservation wage is defined as the wage that achieves indifference in the accept-reject decision. That is, $\bar{w}$ is such that $U(a) = W(\bar{w}, a)$. As we discussed above, the value function of an employed worker is increasing in both arguments. It then follows immediately that the reservation wage should be a function of a. For the purposes of this problem and given the time constraints, you didn't need to go any further than this type of explanation for the reservation wage.

Now that we know the optimal accept-reject decision depends on assets, let's complete the discussion of the unemployed worker's optimal strategy by analyzing his savings problem. We can rewrite (11) as

$$U(a) = \max_{a'} \left\{ u(Ra + z - a') + \beta \int_0^{\bar{w}(a)} U(a')dF(w) + \beta \int_{\bar{w}(a)}^{\infty} W(w, a')]dF(w) \right\} \tag{12}$$

Let's assume differentiability of both U and W in a. Under this assumption, we can characterize the optimal investment policy in (12) by taking FOC and ENV conditions:

$$\begin{aligned} &\text{[FOC]} \qquad u'(c) = \beta \int_0^{\bar{w}(a)} U'(a')dF(w) + \beta \int_{\bar{w}(a)}^{\infty} W_a(w, a')dF(w) \\ &\text{[ENV } U\text{]} \qquad U'(a) = Ru'(c_u) \\ &\text{[ENV } W\text{]} \qquad W_a(w, a) = Ru'(c_e), \end{aligned}$$

where c_u and c_e denote optimal consumption when unemployed and employed, respectively. Combining the above three conditions, we arrive at the following expression which characterizes the optimal saving policy:

$$u'(c) = \beta R \left(\int_0^{\bar{w}(a)} u'(c_u')dF(w) + \int_{\bar{w}(a)}^{\infty} u'(c_e')dF(w) \right) \tag{13}$$