Econ 714: Handout 2 - Solution [1]

1 McCall with savings[2]

Recall the McCall search model: when a worker is unemployed, he receives a constant benefit z, searches for a job, and receives an offer w drawn from a distribution $F(w)$. Wage offers are i.i.d. over time, and if a worker accepts he works forever at that wage w. Now suppose that workers (both employed and unemployed) are able to borrow or save in an asset a yielding fixed gross return R. The flow budget constraint is thus:

$$c_t + a_{t+1} = Ra_t + y_t$$

where here y_t is the workers income, which is either z if he is unemployed or w if he is employed. Assume there is a borrowing constraint $a_t \geq \underline{a}$ which never binds. Workers seek to maximize:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $0 < \beta < 1$ and u is bounded and satisfies $u' > 0$, $u'' < 0$.

1. Write down the Bellman equations characterizing the values of employed and unemployed workers.
2. Characterize an unemployed workers decisions for responding to job offers, consumption, and savings as sharply as you can.

Check Kyle Dempsey's solution of this problem for a great and detailed explanation (duplicated on my web-page). As you can see there, it is enough to derive an Euler equation.

The only part I'm not quite sure about is determination of reservation wage. By timing convention, accept/reject decision is made after a' has been choosen, w_R should be a function of a'. This in turn should be taken into account when taking FOC of value function of the unemployed with respect to a'. One should be careful and use Leibniz's ruls since a' appears both in the integrand $U(a'), W(w, a')$ and limits $w_R(a')$ if the integral.

2 Lucas tree[3]

Suppose that a representative agent has preferences:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

over the single non-storable consumption good (fruit), where $\gamma > 0$. Her endowment of the good is governed by a Markov process with transition function $F(x, x')$.

1. Define a recursive competitive equilibrium with a market in claims to the endowment process (trees).

   Recursive problem of the agent:

$$\begin{aligned} V(a, x) = &\max_{c, a'} \left(u(c) + \beta \mathbb{E}[V(a', x')|x] \right) \\ &\text{s.t. } c + pa' = (p + x)a \end{aligned} \tag{1}$$

   where p is the market price of a claim/asset a.

   Recursive competitive equilibrium is:

[1] By Anton Babkin. This version: February 7, 2016.
[2] Spring 2014 midterm
[3] August 2012 macro prelim

- solution of the agent's problem (1):
  - decision rules $c(a,x)$ and $a'(a,x)$,
  - value function $v(a,x)$;
- prices $p(x)$,

such that markets clear:

- $a'(a,x) = 1$ (assets)
- $c(a,x) = x$ (goods)

2. When the consumer has logarithmic utility, $u(c) = log(c)$, what is the equilibrium price/dividend ratio of a claim to the entire consumption stream? How does it depend on the distribution of consumption growth?

   Start with characterising solution of the agent's problem with Euler equation.

   Substitute c using budget constraint:

$$V(a,x) = \max_{c,a'} (u((p+x)a - pa') + \beta\mathbb{E}[V(a',x')|x])$$

$$\begin{aligned} \text{FOC}(a') &: pu'(c) = \beta\mathbb{E}[V_1(a',x')|x] \\ \text{ENV}(a) &: V_1(a,x) = (p+x)u'(c) \\ \text{EE} &: pu'(c) = \mathbb{E}[\beta u'(c')(p'+x')|x] \end{aligned}$$

   Rewrite EE with time subscripts:

$$p_t = \mathbb{E}_t\beta\frac{u'(c_{t+1})}{u'(c_t)}(p_{t+1} + x_{t+1})$$

   Substitute iteratively, using law of iterated expectations, $\mathbb{E}_t[E_{t+1}[x_{t+2}]] = E_t[x_{t+2}]$:

$$\begin{aligned} p_t &= \mathbb{E}_t[\sum_{s=1}^{T}\beta^s\frac{u'(c_{t+s})}{u'(c_t)}x_{t+s} + \beta^T\frac{u'(c_{t+T})}{u'(c_t)}p_{t+T}] \\ &\to \mathbb{E}_t[\sum_{s=1}^{\infty}\beta^s\frac{u'(c_{t+s})}{u'(c_t)}x_{t+s}] \end{aligned} \tag{2}$$

   where $\lim_{T\to\infty}\mathbb{E}_t\beta^T\frac{u'(c_{t+T})}{u'(c_t)}p_{t+T} = 0$ by transversality condition.

   Now use market clearing $c_t = x_t$ and log utility $u'(c) = 1/c$ to obtain

$$p_t = \mathbb{E}_t[\sum_{s=1}^{\infty}\beta^s x_t] = x_t\sum_{s=1}^{\infty}\beta^s$$

   So $\frac{p_t}{x_t} = \frac{\beta}{1-\beta}$, i.e. price to dividend ratio does not depend on any future information, including distribution function of dividends.

3. Suppose there is news at time t that future consumption will be higher. How will prices respond to this news? How does this depend on the consumers preferences (which could be CRRA, not necessarily log)? Interpret your results.

   Statement about future consumption implies that there is a certain future increase in dividend or it's distribution, since in equilibrium $c = x$.

CRRA utility is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $u'(c) = c^{-\gamma}$.

With market clearing, equation (2) becomes

$$
\begin{aligned}
p_t &= \mathbb{E}_t[\sum_{s=1}^{\infty} \beta^s \frac{x_{t+s}^{-\gamma}}{x_t^{-\gamma}} x_{t+s}] \\
&= x_t^{\gamma} \mathbb{E}_t[\sum_{s=1}^{\infty} \beta^s x_{t+s}^{1-\gamma}]
\end{aligned}
$$

Current price response to future increase in x depends on γ. If $\gamma < 1$, i.e. risk aversion is low, then price increases. Otherwise, risk aversion dominates dividend growth and price falls.

4. Suppose that the endowment process is characterized by lognormal growth. That is, $x_{t+l} = x_t \exp(\xi_{t+1})$, where $\xi_t \sim N(\mu, \sigma^2)$ i.i.d. What is one-period risk free interest rate? How does it depend on the preference parameter γ? Interpret your results.

   Price of an asset that pays $g(x_{t+1})$ in one period is given by:

$$
p_t^g = \mathbb{E}_t[\beta \frac{u'(x_{t+1})}{u'(x_t)} g(x_{t+1})]
$$

   Risk-free discount bond with price q_t pays $g(x_{t+1}) = 1$. With CRRA utility:

$$
\begin{aligned}
q_t &= \mathbb{E}_t[\beta \frac{x_{t+1}^{-\gamma}}{x_t^{-\gamma}} \times 1] \\
&= \mathbb{E}_t[\beta \frac{(x_t e^{\xi_{t+1}})^{-\gamma}}{x_t^{-\gamma}}] \\
&= \beta \mathbb{E}_t e^{-\gamma \xi_{t+1}} \\
&= \beta e^{-\gamma\mu + \gamma^2\sigma^2/2}
\end{aligned}
$$

   Where the last equation is the expectation of log-normal distribution.

   So the gross risk-free interest rate is

$$
R = 1/q = \frac{1}{\beta} e^{\gamma\mu - \gamma^2\sigma^2/2}
$$

5. Consider an option which is bough or sold in period t and entitles the current owner to exercise the right to buy one tree in period t+1 at the fixed price $\bar{p}$ specified at date t. (The buyer may choose not to exercise this option.) Find a formula for the price of this option in terms of the parameters describing preferences and endowments.

   Call option with strike price $\bar{p}$ pays $g(x_{t+1}) = \max\{p(x_{t+1}) - \bar{p}, 0\}$.

   So price of such option must be

$$
p_t^g = \mathbb{E}_t[\beta \frac{x_{t+1}^{-\gamma}}{x_t^{-\gamma}} \max\{p(x_{t+1}) - \bar{p}, 0\}]
$$

   With a known distribution $F(x, x')$, it can be computed using equation (2) for $p(x_{t+1})$.