Econ 714: Handout 3 - Solution [1]

1 Term structure of interest rates

Consider the following environment. The economy has a Lucas tree that issues an uncertain dividend s_t each period, which follows the Markov transition function $Q(s, s')$. Define the risk-free (gross) interest rate between periods t and $t+j$ as R_{jt} (corresponding price $q_{jt} = 1/R_{jt}$), measured in units of the $t+j$ consumption good per the time t consumption good. Assume that all bonds are zero-coupon. Let b_{jt} denote the holdings of the bond with maturity j issued at time t, and let a_t denote holdings of the tree. Suppose the representative agent has time-additive utility such that $u' > 0, u'' < 0$, and $\lim_{c\to 0} u'(c) = \infty$.

1. Assume that the only bonds in the market are of 1- and 2-period maturity, and each is in zero net supply. Write down the agent's Bellman equation and the relevant budget constraint and market clearing conditions (i.e. define an equilibrium).

2. Solve for the prices of all three assets. Does the relationship

$$q_{2t} = q_{1t}\mathbb{E}_t(q_{1t+1})$$

hold? Why or why not? If not, what changes can we make to the model so that it does?

3. The **term structure** is the collection of yields to maturity across dates of maturity for all bonds within a certain class. The **yield to maturity** of a bond is the return earned by an investor who purchases the bond today and holds it to maturity, collecting all coupon payments along the way and the return of principal at the end. Assume that the dividend process is i.i.d. to ease the computation. Compute the yields to maturity and therefore term structure for the risk-free assets in this economy. Under what conditions is the yield curve increasing?

4. BONUS (an easy one): Generalize your answer to part 3 to the case when the economy has bonds with maturities 1,2,...j.

This question is adapted from Section 13.8 of LS.

1. The budget constraint for the representative agent given this environment is

$$c_t + p_t a_{t+1} + q_{1t} b_{1t} + q_{2t} b_{2t} \le (p_t + s_t) a_t + b_{1,t-1} + q_{1t} b_{2,t-1} \quad (1)$$

Notice that on the LHS b_{1t} is indistinguishable from $b_{2,t-1}$, so we can ignore the latter. Similarly, on the RHS $b_{1,t-1}$ is the same as $b_{2,t-2}$.

[1] By Anton Babkin. This version: February 16, 2016.

Assume that we have the initial conditions $b_{10} = b_{20} = 0$. The LHS of (1) describes the expenditures of the agent, taking prices as given. The RHS represents period t wealth. Given this, the Bellman equation is

$$V(a_t, b_{1,t-1}, b_{2,t-1}; s_t) = \max_{a_{t+1}, b_{1t}, b_{2t}} \{u(c_t) + \beta \mathbb{E}[V(a_{t+1}, b_{1t}, b_{2t}; s_{t+1})|s_t]\} \tag{2}$$

$$\text{s. t. } c_t + p_t a_{t+1} + q_{1t} b_{1t} + q_{2t} b_{2t} = (p_t + s_t) a_t + b_{1,t-1} + q_{1t} b_{2,t-1}$$

subject to all the appropriate non-negativity constraints. The relevant market clearing conditions are $c_t = s_t$, $a_{t+1} = 1$, $b_{1t} = b_{2t} = 0$.

2. We can use the usual procedure of taking FOCs and envelope conditions on the Bellman equation (2) and plugging in market clearing conditions to find that:

$$\begin{aligned}
p_t &= \mathbb{E}_t\left[\beta \frac{u'(s_{t+1})}{u'(s_t)}(p_{t+1} + s_{t+1})\right] \\
1/R_{1t} = q_{1t} &= \mathbb{E}_t\left[\beta \frac{u'(s_{t+1})}{u'(s_t)}\right] \\
1/R_{2t} = q_{2t} &= \mathbb{E}_t\left[\beta \frac{u'(s_{t+1})}{u'(s_t)} q_{1,t+1}\right] = \mathbb{E}_t\left[\beta^2 \frac{u'(s_{t+2})}{u'(s_t)}\right]
\end{aligned}$$

The last part of this question asks us to evaluate whether a reasonable replication result holds. But as we will see, the combination of risk aversion and uncertainty in this model will actually make this equation not hold. In order to see this, we can compute:

$$\begin{aligned}
q_{2t} &= \mathbb{E}_t\left[\beta \frac{u'(s_{t+1})}{u'(s_t)} q_{1,t+1}\right] \\
&= \mathbb{E}_t\left[\beta \frac{u'(s_{t+1})}{u'(s_t)}\right] \mathbb{E}_t\left[q_{1,t+1}\right] + Cov_t\left(\beta \frac{u'(s_{t+1})}{u'(s_t)}, q_{1,t+1}\right) \\
&= q_{1t} \mathbb{E}_t\left[q_{1,t+1}\right] + Cov_t\left(\beta \frac{u'(s_{t+1})}{u'(s_t)}, q_{1,t+1}\right) \neq q_{1t} \mathbb{E}_t\left[q_{1,t+1}\right] \text{ in general.}
\end{aligned}$$

Note that under risk neutrality, i.e. linear utility, marginal utility is constant and so the covariance term goes away and the originally proposed result holds.

3. For a zero-coupon bond with purchase price $1/R_{jt}$, the yield to maturity is simply $YTM_{jt} = (R_{jt})^{1/j}$. Recalling that we are now assuming that the dividends are i.i.d. to ease the exposition, we can write the two yields as:

$$\begin{aligned}
YTM_{1t} &= \frac{1}{\beta}\left(\frac{u'(s_t)}{\mathbb{E}(u'(s))}\right) \\
YTM_{2t} &= \frac{1}{\beta}\left(\frac{u'(s_t)}{\mathbb{E}(u'(s))}\right)^{1/2}
\end{aligned}$$

Therefore, it follows that

$$\frac{YTM_{2t}}{YTM_{1t}} = \left(\frac{\mathbb{E}(u'(s))}{u'(s_t)}\right)^{1/2}.$$

Then, in order for the yield curve to be increasing we must have that $\mathbb{E}(u'(s)) > u'(s_t)$. When consumption is high, marginal utility is low, while expected future consumption is lower. So everybody wants to save more in order to smooth their consumption. Increased demand for bonds increases their price and reduces interest rates, return to savings falls on both 1 and 2 year bonds. But next period consumption is expected to revert to it's mean, and interest rates are expected to be higher. Saving in 1 year bonds can be reinvested then for a higher return, so demand for 1 year bonds today is higher than for 2 year bonds, reducing their return more (in annualized terms, or YTM).

4. Using the same approach that we used in parts 2 and 3, we can find that the j-period ahead risk-free rate is given by

$$1/R_{jt} = \mathbb{E}_t\left[\beta^j \frac{u'(s_{t+j})}{u'(s_t)}\right],$$

and so the corresponding yield to maturity is given by

$$YTM_{jt} = \frac{1}{\beta}\left[\frac{u'(s_t)}{E(u'(s))}\right]^{1/j}$$

By iterating forward on the above expression, we can solve for the relationships between yields of all maturities. Then, in general, for $k > j$,

$$\frac{YTM_{kt}}{YTM_{jt}} = \left(\frac{\mathbb{E}(u'(s))}{u'(s_t)}\right)^{(k-j)/kj},$$

and so the criteria for an increasing yield curve are the same as above.

2 Burrowed are?[2]

An economy consists of two types of consumens indexed by $i = 1, 2$. There is one nonstorable consumption good. Let (e_t^i, c_t^i) be the endowment, consumption pair for consumer i in period t. Both consumers have preferences ordered by

$$U^i = \mathbb{E} \sum_{t=0}^{\infty} \beta^t \log(c_t^i)$$

The endowment streams of the two consumers are governed by two independent 2-state Markov chains $s_t \in \{0, 1\}$ with transition matrix P_s and $a_t \in \{0, 1\}$ with transition matrix P_a, where $P_{s,12} = Pr\{s_{t+1} = 1 | s_t = 0\}$ and so on. Suppose that each chain is stationary and that the initial states (s_0, a_0) are drawn from the stationary distribution, so that $Pr\{s_0 = 0\} = \overline{\pi}_s$ and $Pr\{a_0 = 0\} = \overline{\pi}_a$. Consumer 1 has endowment $e_t^1 = a_t + s_t$, while consumer 2 has endowment $e_t^2 = a_t + 1 - s_t$.

(a) Write out explicitly the preferences U^i of a consumer in terms of the history of the states and their associated probabilities.

(b) Define a competitive equilibrium for this economy.

(c) Characterize the competitive equilibrium for this economy, calculating the prices of all Arrow-Debreu securities. How does the allocation vary across the (s_t, a_t) states? Across time? Across consumers?

(d) Now consider an economy with sequential trading in Arrow securities, one-period ahread claims to contingent consumption. How many Arrow securities are there? Compute their prices in the special case $\beta = 0.95$, $P_{s,11} = 0.9$, $P_{s,22} = 0.8$, $P_{a,11} = 0.8$, $P_{a,22} = 0.7$.

(e) Using these same parameters, in each state what is the price of a one-period ahead riskless claim to one unit of consumption?

You can check details in the solution of Problem set 1, but here is a general overview of the algoritm.

Solution relies on a few fundamental results, proven in LS Chapter 8.

- For a particular choice of prices date-0 trade (Arrow-Debreu) market equilibrium allocation is the same as in secquential (Arrow) market equilibrium.
- AD equilibrium is Pareto efficient.
- Solution to a social planner's problem is Pareto efficient.

Negishi algorithm uses these results to solve for competitive equilibrium:

[2]Spring 2016 problem set

1. Solve social planner's problem with a set of Pareto weights. Solution is an Pareto efficient consumption sequence that depends on chosen weights.

2. We know that *one* of all possible Pareto efficient allocations coincides with AD market equilibrium. To select such allocation, we need to pick particular Pareto weights which will satisfy a feature of markets, budget constraint, and be consistent with household optimization. In addition, one price can be normalized to 1 (typically, price at time 0 and state 0. Once we found such weights, we can compute AD equilibrium.

3. Sequential markets equilibrium has the same allocation as AD equilibrium and prices $q^t_{t+1} = q^0_{t+1}/q^0_t$.