Econ 714: Handout 4 - Solution [1]

1 Phase diagrams[2]

Consider a version of the (deterministic) optimal growth model with government. There is an exogenous stream of government purchases $\{G_t\}$ that the planner takes as given. The household does not value government purchases, but they must be funded with real resourses. So the planner chooses the allocation of consumption c_t and capital k_{t+1} to maximize the household utility (over consumption, with labor supplied inelastically) subject to resource contraint:

$$k_{t+1} + c_t + G_t = (1 - \delta)k_t + f(k_t)$$

1. Suppose that government purchases are constant at $G_t = G$. How does the introduction of government spending affect the steady state levels of consumption and capital, relative to the case where $G = 0$?

2. Suppose that initially the economy is in a steady state with $G_t = G$, then there is a once-and-for-all unforeseen increase in purchases to a new higher level $G' > G$. What happens to consumption and capital immediately upon the impact of the change and in the long run?

3. Suppose that initially the economy is in a steady state with $G_t = G$, then at date T there is announcement that at the future date $T' > T$ purchases will increase to a new higher level $G' > G$ and remain there. What happens to consumption and capital at T, the date of the announcement? What happens between T and T'? What happens at T'?

Social planner's problem is:

$$V(k) = \max_{c,k'}(u(c) + \beta V(k'))$$

subject to resource contraint.

Solution satisfies two conditions, Euler equation and resource constraint (capital law of motion):

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta(1 + f'(k_{t+1}) - \delta) \tag{EE}$$

$$k_{t+1} - k_t = f(k_t) - \delta k_t - G_t - c_t \tag{RC}$$

This system of first-order nonlinear difference equations can be analysed graphically in a phase diagram, shown in Figure 1. Saddle path and four non-stable trajectories are shown with arrows.

1. (EE) can be solved for a steady level of capital $\bar{k}$ by setting $c_t = c_{t+1}$, and then $\bar{c}$ can be found from (RC) by setting $k_t = k_{t+1}$. $\bar{k}$ does not depend on G, but positive government pending will crowd out consumption one-to-one as can be seen from the (RC).

[1] By Anton Babkin. This version: February 22, 2016.

[2] Fall 2014 problem set

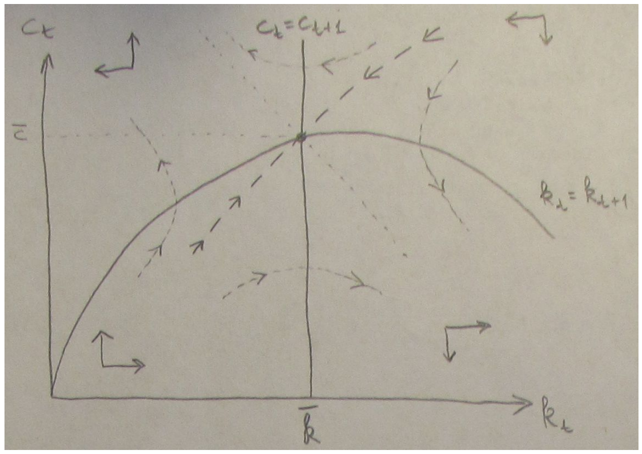

Figure 1: Phase diagram of the optimal growth model with $G_t = 0$.

2. Increase in G shifts the $k_t = k_{t+1}$ curve down on the phase diagram. New steady state $\bar{\bar{k}} = \bar{k}$ and $\bar{\bar{c}} < \bar{c}$. Consumption adjusts immediately to the new level and capital does not change. See Figure 2 for a phase diagram and trajectories of G_t, k_t and c_t.

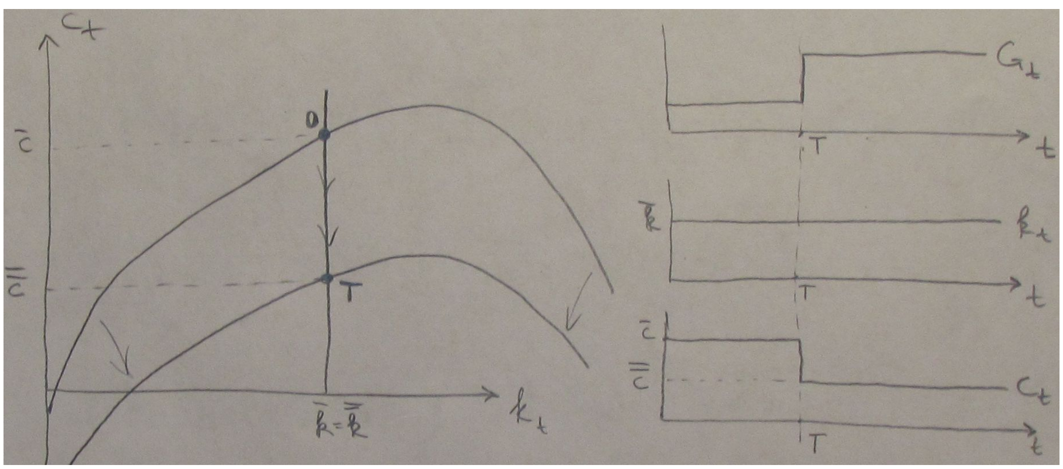

Figure 2: Phase diagram and time series plots of trajectories for an unticipated increase in G_t at $t = T$.

3. Old and new steady states are like in part 2, but transitional dynamics is now different. As information about future increase in G becomes known, households start accumulating additional capital above steady state level to build up a buffer stock that will be used to smooth consumption trajectory. To build more capital, households have to save more and consume less. So there is an immediate drop in consumption, although not as big as in part 2. Between T and T' the dynamic system is on a non-stable

trajectory, but at T' it must be exactly on the saddle path of the new equilibrium, so it then converges gradually to the new steady state. See Figure 3 for a phase diagram and trajectories of G_t, k_t and c_t.

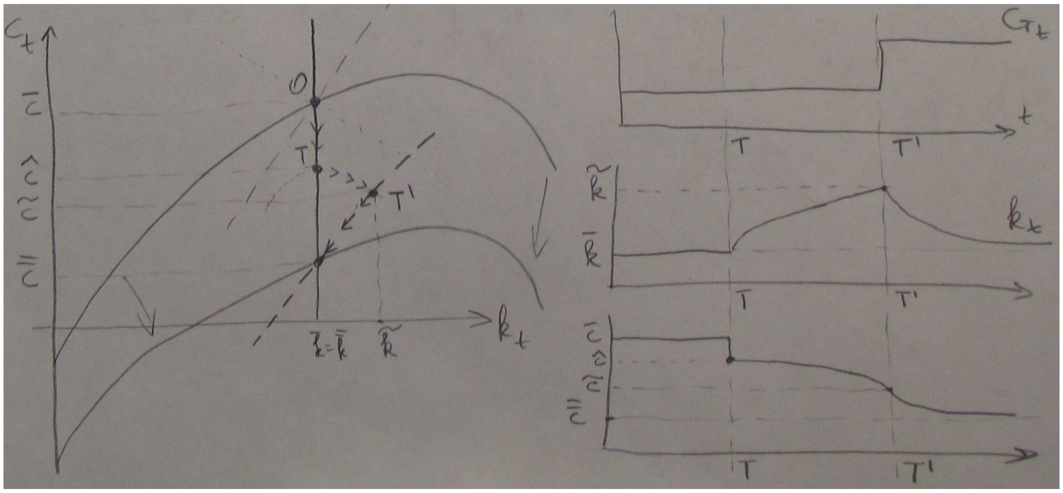

Figure 3: Phase diagram and time series plots of trajectories for an increase in G_t at $t = T'$, announced at $t = T$.

2 Ramsey model[3]

Consider the following growth model. There is a representative household whose utility function is $\sum_{t=0}^{\infty} \beta^t (log(c_t) - \frac{1}{2}n_t^2)$, where c_t is consumption and n_t is labor supply. The resource constraint is: $c_t + g_t = n_t$, where g_t is government spending given by:

$$g_t = \begin{cases} 0 & \text{for } t \neq 10 \\ \bar{g} & \text{for} t = 10 \end{cases}$$

where $\bar{g} > 0$. The government takes the g_t sequence above as given and uses linear taxes on labor income and debt to finance it.

1. Define an equilibrium for this model economy.
2. Formulate the Ramsey problem for the government.
3. Draw a time series plot of the optimal labor income tax rates for period $t = 0$ through $t = 20$.

Check LS 15.6 for a more detailed explanation of the primal approach.

1. Assume that the borrowing limit is such that it never binds. An equilibrium is a sequence of $\{c_t, n_t, b_t, N_t, \pi_t, w_t, q_t\}_{t=0}^{t=\infty}$ given a fiscal policy, $\{\tau_t^n, g_t, B_t\}_{t=0}^{t=\infty}$ such that

[3] August 2013 prelim

(a) $\{c_t, n_t, b_t\}_{t=0}^{\infty}$ solves the representative HH's problem given $\{q_t, w_t, \pi_t\}_{t=0}^{\infty}$,

$$\begin{aligned}
&\max_{\{c_t,n_t,b_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (log(c_t) - \frac{1}{2}n_t^2) \\
\text{s.t. } & c_t + q_t b_{t+1} = w_t n_t (1-\tau_t^n) + b_t + \pi_t \quad \text{for all } t, \\
& b_0 = \bar{b}_0, \\
& c_t, n_t, 1-n_t \geq 0 \quad \text{for all } t,
\end{aligned}$$

(b) $\{N_t\}$ solves the representative firm's problem given $\{w_t\}$ for all t,

$$\begin{aligned}
&\max_{N_t} N_t - w_t N_t \\
\text{s.t. } & N_t \geq 0
\end{aligned}$$

(c) Markets clear:

$$\begin{aligned}
& b_t = B_t \quad \text{for all } t, \\
& c_t + g_t = N_t \quad \text{for all } t, \\
& n_t = N_t \quad \text{for all } t,
\end{aligned}$$

(d) Government budget constraint holds:

$$g_t + B_t = \tau_t^n w_t n_t + q_t B_{t+1}$$

Government budget constraint is redundant by Walras law: it is satisfied as long as HH budget and market clearing conditions are (check).

We can also rewrite the problem with lifetime budget constraint and trades happening at $t = 0$. Substitute iteratively b_{t+i} in the household and denote $q_t^0 = \Pi_{s=1}^t q_s$ (and $q_0^0 = 1$) to get:

$$\sum_{t=0}^{\infty} q_t^0 c_t = \sum_{t=0}^{\infty} q_t^0 w_t n_t (1-\tau_t^n) + \bar{b}_0$$

2. The Ramsey problem for the government is to chose the fiscal policy (τ_t^l, B_t) for a given $\{g_t\}$ such that corresponding competitive market equilibrium attains highest possible HH utility.

   The *primal approach* to the Ramsey problem solution is to find allocations subject to implementability constraints, i.e. such allocations solve HH problem and satisfy market clearing conditions. The key insight is to reduce dimensionality of the optimization problem by substituting out prices and taxes.

   With η attached as a Lagrange multiplier to the budget constraint, HH first order conditions are:

$$\begin{aligned}
&[c_t] : \beta^t u_c(t) - \eta q_t^0 = 0 \\
&[n_t] : \beta^t u_n(t) + \eta q_t^0 w_t (1-\tau_t^n) = 0
\end{aligned}$$

From the condition $[c_t]$ and $q_0^0 = 1$, find $\eta = u_c(0)$. Use this with the $[c_t]$ condition to get $q_t^0 = \beta^t \frac{u_c(t)}{u_c(0)}$. Divide $[n_t]$ by $[c_t]$ to get $w_t(1-\tau_t^n) = -\frac{u_n(t)}{u_c(t)}$. Together with firm's problem solution $w_t = 1$, all prices and taxes can now be expressed in terms of allocations c_t, n_t.

Can now rewrite life-time budget constraint as

$$\sum_{t=0}^{\infty} \beta^t (u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t) = u_c(c_0, n_0)\bar{b}_0$$

The Ramsey problem reduces to

$$\max_{\{c_t,n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (log(c_t) - \frac{1}{2}n_t^2)$$

$$\text{s.t.} \sum_{t=0}^{\infty} \beta^t (u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t) = u_c(c_0, n_0)\bar{b}_0$$

$$c_t + g_t = n_t \quad \forall t$$

3. The Lagrangian of the Ramsey problem:

$$\begin{aligned} L &= \sum_{t=0}^{\infty} \beta^t (log(c_t) - \frac{1}{2}n_t^2) + \sum_{t=0}^{\infty} \lambda_t (n_t - c_t - g_t) + \mu(\sum_{t=0}^{\infty} \beta^t (u_c(c_t, n_t)c_t + u_n(c_t, n_t)n_t) - u_c(c_0, n_0)\bar{b}_0) \\ &= \sum_{t=0}^{\infty} \beta^t (log(c_t) - \frac{1}{2}n_t^2) + \sum_{t=0}^{\infty} \lambda_t (n_t - c_t - g_t) + \mu(\sum_{t=0}^{\infty} \beta^t (1 - n_t^2) - \frac{\bar{b}_0}{c_0}) \end{aligned}$$

First-order conditions:

$$[c_t] : \lambda_t = \frac{\beta^t}{c_t}$$

$$[n_t] : \lambda_t = \beta^t n_t (1 + 2\mu)$$

Combine the FOCs above to get $c_t n_t = \frac{1}{1+2\mu}$. Remember from HH FOCs $w_t(1-\tau_t^n) = -\frac{u_n(t)}{u_c(t)} = c_t n_t$ and $w_t = 1$, so the optimal labor income tax $\tau_t^n = \frac{2\mu}{1+2\mu}$, constant $\forall t$. Also from $c_t + g_t = n_t$ it follows that $c_t = n_t = \sqrt{\frac{1}{1+2\mu}}$ for $t \neq 10$.

This is an illustration of tax smoothing. Instead of taxing large amount when government spending is made, debt is used to redistribute tax distortions over time. With the same tax revenues in all periods before and after time $t = 10$, the optimal debt policy is as follows: In each period $t = 0, 1, ..., 9$, the government runs a surplus, using it to buy bonds issued by the private sector. In period $t = 10$, the expenditure $\bar{g}$ is met by selling all the bonds, levying the same tax on current labor income, and issuing new bonds that thereafter rolled over forever. Interest payments on that constant outstanding government debt are financed with constant tax revenue.