Econ 714: Handout 5 - Solution [1]

1 Optimal taxation with private information[2]

The model economy has two periods and a unit measure of agents. Each angent is endowed with y_1 units of the single consumption good in period 1. Consumption can be stored from one period to the next.

In period 2, agents can exert effort to generate consumption. Measure p_H of the agents are highly skilled, and one unit of their effort generates θ_H units of consumption. Measure p_L are low-skilled. For them, one unit of effort generates θ_L units of consumption, where $\theta_L < \theta_H$. $p_H + p_L = 1$.

Agents' utility function is

$$U(c_1, c_2, l_2) = u(c_1) + \mathbb{E}[u(c_2) - v(l_2)]$$

where c_t is consumption in period t and l_t is effort in period t. $u' > 0, u'' < 0, v' > 0, v'' > 0$.

Timing is as follows. In period 1 agents choose consumption c_1 and savings S. In the beginning of period 2 idiosyncratic productivity θ is realized, with $Pr(\theta = \theta_H) = p_H$ and $Pr(\theta = \theta_L) = p_L$. Then the agent chooses c_2 and l_2. Productivity and effort are private invormation, but savings and output $y = \theta l$ are publicly observed.

1. Characterize first-best allocation. Explain why it is not attainable under private information.

   When all information is observable, social planner can choose allocations conditional on agent's type. Denote by x^i the choice of variable x by an agent of realized type θ_i. Note that ex ante utility is maximized, and second period variables are chosen for every type before realization of uncertainty.

   Social planner's problem:

$$\max_{c_1, S, c_2^i, l_2^i} u(c_1) + \sum_i p_i(u(c_2^i) - v(l_2^i))$$
$$\text{s.t. } c_1 + S = y_1$$
$$p_L c_2^L + p_H c_2^H = S + p_L \theta_L l_2^L + p_H \theta_H l_2^H$$

   Attach Lagrange multiplier λ to the combined resource constraint (substitute out S) $c_1 + p_L c_2^L + p_H c_2^H = y_1 + p_L \theta_L l_2^L + p_H \theta_H l_2^H$ and find FOCs:

[1] By Anton Babkin. This version: February 29, 2016.

[2] Adapted from Kocherlakota (2004) Wedges and Taxes, AER Papers and Proceedings.

$$u'(c_1) = \lambda$$
$$p_i u'(c_2^i) = p_i \lambda$$
$$p_i v'(l_2^i) = p_i \theta_i \lambda$$

Summing the second equation over i and equating to the first, obtain Euler equation $u'(c_1) = \mathbb{E}(u'(c_2^i))$. Divide second by p_i and equate with the first to get $c_1 = c_2^L = c_2^H$: consumption is equalized across time and across states.

Use FOC in l_2^i for $i = L, H$ and divide to get $\frac{v'(l_2^L)}{v'(l_2^H)} = \frac{\theta_L}{\theta_H}$. Then $\theta_L < \theta_H$ implies $l_2^L < l_2^H$: high productivity agents put more effort even though their consumption level is the same. Nevertheless, it is optimal ex ante, since agents don't know what type they will be in period 2.

With private information, types are not observable to the planner, and θ_H types will claim that they are θ_L to exert less effort. So this first-best allocation is not compatible with incentives under private information.

2. Formulate social planner's problem with private information. Characterize socially optimal allocation: derive the Euler equation and show that there is a wedge between intertemporal MRS and MRT.

   Under private information, allocations can only be conditional on agent's reported types. To guarantee that types would be reported truthfully, allocations should be such that agents have no incentives to pretend that they are a different type. Such allocations must satisfy *incentive compatibility* constraints. The full social planner's problem becomes:

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$$\max_{c_1, S, c_2^i, l_2^i} u(c_1) + \sum_i p_i(u(c_2^i) - v(l_2^i))$$
$$\text{s.t. } c_1 + p_L c_2^L + p_H c_2^H = y_1 + p_L \theta_L l_2^L + p_H \theta_H l_2^H$$
$$u(c_2^H) - v(y_2^H/\theta_H) \geq u(c_2^L) - v(y_2^L/\theta_H) \qquad (IC_H)$$
$$u(c_2^L) - v(y_2^L/\theta_L) \geq u(c_2^H) - v(y_2^H/\theta_L) \qquad (IC_L)$$

   It can be proved (and we will just assume here) that (IC_L) does not bind in equilibrium.

   (IC_H) always binds in equilibrium. To prove, assume that it's not. Then it is possible to increase c_2^L and decrease c_2^H so that resource constraint is still satisfied. (IC_L) still holds, as LHS goes up and RHS goes down. And since by assumption (IC_H) holds with strict inequality, we can always choose sufficiently small change in c_2^H, c_2^L so that (IC_H) still holds. But such alternative allocation is strictly better for aggregate welfare because

$c_2^L < c_2^H$ and $u'(c_2^L) > u'(c_2^H)$. So we reached a contradiction, and hence (IC_H) must be binding.

Rewrite the problem:

$$\max_{c_1,S,c_2^i,l_2^i} u(c_1) + \sum_i p_i(u(c_2^i) - v(l_2^i))$$
$$\text{s.t. } c_1 + p_L c_2^L + p_H c_2^H = y_1 + p_L\theta_L l_2^L + p_H\theta_H l_2^H$$
$$u(c_2^H) - v(y_2^H/\theta_H) = u(c_2^L) - v(y_2^L/\theta_H)$$

Attach multipliers λ and μ to constraints. To derive Euler equation, we only need FOCs on consumption.

$$u'(c_1) = \lambda$$
$$p_L u'(c_2^L) - p_L\lambda - \mu u'(c_2^L) = 0$$
$$p_H u'(c_2^H) - p_H\lambda + \mu u'(c_2^H) = 0$$

Substitute λ from the first equation and solve the last two equations for μ, obtain:

$$\frac{p_L u'(c_2^L) - p_L u'(c_1)}{u'(c_2^L)} = \frac{p_H u'(c_1) - p_H u'(c_2^H)}{u'(c_2^H)}$$

Solve for $u'(c_1)$:

$$u'(c_1) = \frac{1}{p_L \frac{1}{u'(c_2^L)} + p_H \frac{1}{u'(c_2^H)}}$$

This is the "reciprocal" Euler equation: $u'(c_1) = \frac{1}{\mathbb{E}[1/u'(c_2^i)]}$.

By Jensen's inequality, $\mathbb{E}[1/u'(c_2^i)] > 1/\mathbb{E}u'(c_2^i)$, and from Euler equation $u'(c_1) < \mathbb{E}u'(c_2^i)$. In this equilibrium $MRS \neq MRT$, because there is an information friction.

3. Formulate agent's problem in a decentralized environment, where government imposes a tax $\tau(S, y_2)$ in period 2. Assume that $p_L = p_H = 1/2$. Show that socially optimal marginal tax on savings $\tau_S(S, y_2)$ depends on y_2 and $\tau_S(S, y_2^H) < \tau_S(S, y_2^L)$.

   To decentralize the second-best allocation means to find an appropriate tax schedule, such that socially optimal allocation will be choosen by utility-maximizing agents.

Let the tax schedule be $\tau(S, y_2^i)$: it only depends on observables (can't condition taxes on θ_i or l_2^i) and can be a non-linear function.

Agent's problem is:

$$\max_{c_1,S,c_2^i,l_2^i} u(c_1) + \sum_i p_i(u(c_2^i) - v(l_2^i))$$
$$\text{s.t. } c_1 + S = y_1$$
$$c_2^i = S + y_2^i - \tau(S, y_2^i) \quad \forall i$$
$$y_2^i = \theta_i l_2^i$$

Substitute budget constraints in the objective function:

$$\max_{S,y_2^i} u(y_1 - S) + \sum_i p_i[u(S + y_2^i - \tau(S, y_2^i)) - v(y_2^i/\theta_i)]$$

FOC in S gives Euler equation:

$$u'(c_1) = \sum_i p_i u'(c_2^i)(1 - \tau_S(S, y_2^i))$$

For agent's solution to coincide with social optimum, allocation should also satisfy (IC_H) with equality. In other words, θ_H-type agent should be indifferent between allocations (c_2^L, y_2^L) and (c_2^H, y_2^H). If the agent chooses (c_2^L, y_2^L) for any i, his EE becomes

$$u'(c_1) = \sum_i p_i u'(c_2^L)(1 - \tau_S(S, y_2^L)) = u'(c_2^L)(1 - \tau_S(S, y_2^L))$$

If he chooses different allocations for $i = L, H$, then EE is

$$u'(c_1) = p_L u'(c_2^L)(1 - \tau_S(S, y_2^L)) + p_H u'(c_2^H)(1 - \tau_S(S, y_2^H))$$

Eliminate $u'(c_1)$ and derive:

$$u'(c_2^L)(1 - \tau_S(S, y_2^L)) = p_L u'(c_2^L)(1 - \tau_S(S, y_2^L)) + p_H u'(c_2^H)(1 - \tau_S(S, y_2^H))$$
$$(1 - p_L)u'(c_2^L)(1 - \tau_S(S, y_2^L)) = p_H u'(c_2^H)(1 - \tau_S(S, y_2^H))$$
$$\frac{1 - \tau_S(S, y_2^L)}{1 - \tau_S(S, y_2^H)} = \frac{p_H}{1 - p_L} \frac{u'(c_2^H)}{u'(c_2^L)}$$

With $p_i = 1/2$ and $c_2^L < c_2^H$, RHS < 1 and $\tau_S(S, y_2^L) > \tau_S(S, y_2^H)$. This means that marginal tax on savings (wealth) depends on the labor income in the second period and is higher for low-productivity agents.

2 Cash-in-advance[3]

Consider a cash-in-advance model in which there are two types of goods: c_1 requires money M_t to purchase, while c_2 can be purchased on credit. The two goods are technologically equivalent, as the endowment e_t can be converted one-for-one into either of them, so $e_t = c_{1t} + c_{2t}$. Suppose that e_t follows a Markov process with transition density $Q(e'|e)$. A representative agent in this economy thus solves:

$$\max_{\{c_{1t}, c_{2t}, M_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{2t})$$

subject to the budget constraint:

$$P_t c_{1t} + P_t c_{2t} = P_t e_t + M_t - M_{t+1},$$

and the cash-in-advance constraint:

$$P_t c_{1t} \leq M_t.$$

1. Write down the Bellman equation for the representative household and find the optimality conditions.

   Recursive formulation:

$$V(M, e) = \max_{c_1, c_2, M'} u(c_1, c_2) + \beta \mathbb{E}[V(M', e')|e]$$
$$\text{s.t. } c_1 + c_2 = e + \frac{M - M'}{P}$$
$$Pc_1 \leq M$$

   With Lagrangian multipliers on constraints:

$$V(M, e) = \max_{c_1, c_2, M'} u(c_1, c_2) + \eta(e + \frac{M - M'}{P} - c_1 - c_2) + \lambda(M - Pc_1) + \beta \mathbb{E}[V(M', e')|e]$$

   Taking first order conditions:

$$[c_1] : u_1(c_1, c_2) = \eta + \lambda P$$
$$[c_2] : u_2(c_1, c_2) = \eta$$
$$[M'] : \beta \mathbb{E}[V_1(M', e')|e] = \eta / P$$

   Envelope condition:

$$[M] : V_1(M, e) = \eta / P + \lambda$$

[3] Spring 2013 problem set.

Combine to obtain Euler equation:

$$\frac{\eta}{P} = \beta\left(\frac{\eta'}{P'} + \lambda'\right)$$

If CIA constraint is not binding, $\lambda = 0$:

$$u_1(c_1, c_2) = u_2(c_1, c_2), \quad \frac{u_1(c_1, c_2)}{P} = \beta\frac{u_1(c_1', c_2')}{P'}$$

If it is binding, $\lambda > 0$:

$$u_1(c_1, c_2) > u_2(c_1, c_2), \quad \frac{u_2(c_1, c_2)}{P} = \beta\frac{u_1(c_1', c_2')}{P'} > \beta\frac{u_2(c_1', c_2')}{P'}$$

2. Consider a steady state equilibrium in which the endowment is constant $e_t = e$, the money supply grows at a constant rate: $M_{t+1} = \mu M_t$, and real balances M_t/P_t are constant. What is the minimal level of μ that will support a steady state monetary equilibrium? Is such equilibrium Pareto efficient?

   Steady state conditions imply $P' = \mu P$, $u_2(c_1, c_2) = const$ and $\eta = \eta'$. The Euler equation becomes:

$$\mu\eta = \beta(\eta' + \lambda' P')$$
$$\mu = \beta + \lambda'\beta P'/\eta$$

   Lagrange multiplier $\lambda' \geq 0$, so the minimal level of μ consistent with household optimization is $\mu = \beta$ with $\lambda = 0$. Such equilibrium is Pareto optimal, since it satisfies the Friedman rule: real rate of return on money is equal to a risk-free rate of return: $P/P' = 1/\mu = 1/\beta$. CIA constraint is "barely" binding ($\lambda = 0$, but money is held in equilibrium) and saving in the form of money is as efficient as any other asset.