Econ 714: Handout 10 - Solution [1]

1 Investment with adjustment costs and taxation[2]

Firm owns productive capital K_t that generates output $F(K_t)$ ($F_K > 0, F_{KK} \leq 0$) and evolves according to $K_{t+1} = (1-\delta)K_t + I_t$. Output can be transformed into investment goods I_t one for one, but investment entails convex adjustment costs of $\Psi(I_t, K_t)$: $\Psi_I > 0, \Psi_{II} > 0, \Psi_K < 0, \Psi_{KK} > 0$ and $\Psi_I(\delta K, K) = 0$, i.e. marginal adjustment cost is zero when investment just replaces depreciating capital. One commonly used functional form is $\Psi(I, K) = \frac{psi_0}{2K}(I - \delta K)^2$.

Corporate profits are subject to taxation characterized by the following rules:

- Operating profit is taxed at rate τ.
- Depreciation allowance. Capital expenditures can be deducted from taxable profit at depreciation schedule D_s, where $s = 0, 1, 2, ...$ is the number of periods since the capital was installed. Assume that D_s follows a simple linear rule: every period a constant fraction δ of the *initial* value of capital can be deducted, i.e. for tax purposes capital fully depreciates after $1/\delta$ periods.
- Investment tax credit: A fraction κ of capital expenditures can be subtracted from the tax bill immediately.
- Assume that the above rules symmetrically apply if before-tax profit is negative, in which case firm gets a refund.

Firm starts with initial level of capital K_0 and is choosing optimal investment policy to maximize present value of after-tax profits, discounted at interest rate r, $V(K_0)$.

1. Formulate firm's decision problem. Pay attention to all the tax rules.

$$V(K_0) = \max_{\{I_t, K_{t+1}\}} \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[(1-\tau)\left(F(K_t) - \Psi(I_t, K_t)\right) - (1-\kappa)I_t + \tau \sum_{s=0}^{\infty} D_s I_{t-s} \right]$$
$$\text{s.t. } K_{t+1} = (1-\delta)K_t + I_t$$

2. Denote the shadow value of capital by q_t. Write down the Lagrangian and characterize firm's optimal investment policy.

   We can rearrange summation as

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \tau \sum_{s=0}^{\infty} D_s I_{t-s} = \sum_{t=0}^{\infty} z I_t + A_0,$$

   where $z = \tau \sum_{s=0}^{\infty} \frac{D_s}{(1+r)^s}$ and $A_0 = \tau \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \tau \sum_{s=0}^{\infty} D_{-1-s} I_{-1-s}$. Note that A_0 is predetermined when decision is made at $t = 0$.

[1] By Anton Babkin. This version: April 17, 2016.

[2] Adapted from Hayashi (1982) "Tobin's marginal q and average q: a neoclassical interpretation", Econometrica.

Rewrite the problem as

$$V(K_0) = \max_{\{I_t, K_{t+1}\}} \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[(1-\tau)\left(F(K_t) - \Psi(I_t, K_t)\right) - (1-\kappa-z)I_t\right] + A_0$$
$$\text{s.t. } K_{t+1} = (1-\delta)K_t + I_t$$

Lagrangian:

$$L = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[(1-\tau)\left(F(K_t) - \Psi(I_t, K_t)\right) - (1-\kappa-z)I_t + q_t\left((1-\delta)K_t + I_t - K_{t+1}\right)\right] + A_0$$

Taking first order conditions

$$[I_t] : q_t - (1-\kappa-z) = (1-\tau)\Psi_I(I_t, K_t)$$
$$[K_{t+1}] : (1+r)q_t = (1-\tau)(F_K(K_{t+1}) - \Psi_K(I_{t+1}, K_{t+1})) + q_{t+1}(1-\delta)$$

FOC in $[I_t]$ can be used to solve for optimal I_t as a function of current K_t and q_t.

3. Assume that firm starts at the steady state. Use phase diagram to describe firm behavior after an unanticipated policy change that allows to depreciate capital for tax purposes at a faster rate $\hat{\delta} > \delta$ (depreciation rule is still linear, and physical depreciation is not affected).

We will be building phase diagram in (K_t, q_t) space.

Rewrite FOCs as

$$K_{t+1} - K_t = \Psi_I^{-1}\left(\frac{q_t - (1-\kappa-z)}{1-\tau}, K_t\right) - \delta K_t$$
$$q_{t+1} - q_t = \delta q_{t+1} + r q_t - (1-\tau)(F_K(K_{t+1}) - \Psi_K(I_{t+1}, K_{t+1}))$$

The expression for the $\Delta K = 0$ isocline is simply a horizontal line $q_t = 1-\kappa-z$ because by assumption $\Psi_I(\delta K, K) = 0$. Ψ_I is increasing in first argument since $\Psi_{II} > 0$, so Ψ_I^{-1} is increasing too. Then if q_t is above $\Delta K = 0$ line $K_{t+1} - K_t > 0$, and below the line $K_{t+1} - K_t < 0$.

The expression for the $\Delta q = 0$ isocline is $q_t = \frac{1}{\delta+r}(1-\tau)(F_K(K_{t+1}) - \Psi_K(I_{t+1}, K_{t+1}))$. It is a downward sloping line because by assumptions $F_{KK} \leq 0$ and $\Psi_{KK} > 0$. When q_t and K_t are big $q_{t+1} - q_t > 0$, and below the line $q_{t+1} - q_t < 0$.

Remember expression for present value of depreciation allowances: $z = \tau \sum_{s=0}^{\infty} \frac{D_s}{(1+r)^s}$. If D_s for smaller capital age s becomes larger, z increases.

Dynamics of the system after the shock is shown in Figure 1. q_t immediately jumps down to the new saddle path, and system gradually converges to the new steady state with higher level of capital.

4. (Hayashi theorem). Show that if $F(K)$ and $\Psi(I, K)$ are linearly homogenuous, then Tobin's marginal q and average $Q \equiv V/K$ are related as $q = Q + \hat{A}$, where $\hat{A}$ is a constant.

Rewrite the FOC in $[K_{t+1}]$ multiplied by K_{t+1}:

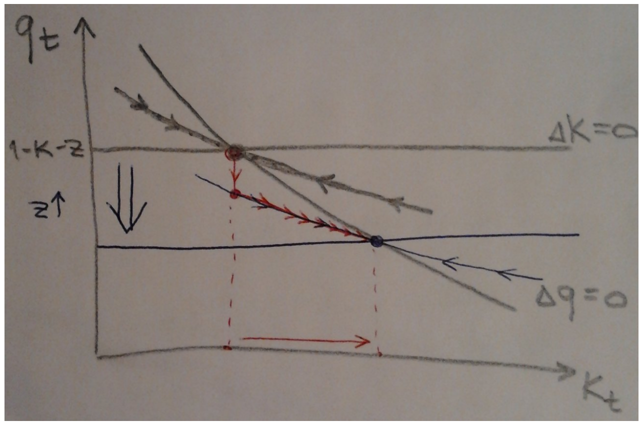

Figure 1: Phase diagram of the increase in z.

$$(1+r)q_t K_{t+1} = (1-\tau)(F_K(t+1)K_{t+1} - \Psi_K(t+1)K_{t+1}) + q_{t+1}K_{t+1}(1-\delta)$$

By Euler theorem $F(K) = F_K K$ and $\Psi(I,K) = \Psi_I I + \Psi_K K$. Use this to manipulate the above equation:

$$(1+r)q_t K_{t+1} = (1-\tau)(F(t+1) - (\Psi(t+1) - \Psi_I(t+1)I_{t+1}) + q_{t+1}K_{t+1}(1-\delta)$$
$$(1+r)q_t K_{t+1} = (1-\tau)(F(t+1) - \Psi(t+1)) + (1-\tau)\Psi_I(t+1)I_{t+1} + q_{t+1}K_{t+1}(1-\delta)$$

Substitute $(1-\tau)\Psi_I(t+1)$ from the FOC in $[I_t]$:

$$(1+r)q_t K_{t+1} = (1-\tau)(F(t+1) - \Psi(t+1)) + (q_{t+1} - (1-\kappa-z))I_{t+1} + q_{t+1}K_{t+1}(1-\delta)$$
$$(1+r)q_t K_{t+1} = (1-\tau)(F(t+1) - \Psi(t+1)) - (1-\kappa-z))I_{t+1} + q_{t+1}(I_{t+1} + K_{t+1}(1-\delta))$$
$$(1+r)q_t K_{t+1} = (1-\tau)(F(t+1) - \Psi(t+1)) - (1-\kappa-z))I_{t+1} + q_{t+1}K_{t+1}$$
$$q_t K_{t+1} = \frac{1}{1+r}\left[(1-\tau)(F(t+1) - \Psi(t+1)) - (1-\kappa-z))I_{t+1} + q_{t+1}K_{t+2}\right]$$

Substitute $q_t K_{t+1}$ forward recursively:

$$q_0 K_1 = \sum_{t=1}^{\infty} \frac{1}{(1+r)^t}\left[(1-\tau)\left(F(K_t) - \Psi(I_t, K_t)\right) - (1-\kappa-z)I_t\right] + A_1 + \lim_{T\to\infty}\frac{1}{(1+r)^T} q_T K_{T+1}$$
$$= \frac{1}{(1+r)} V(K_1) - A_1$$

So $q_0 = \frac{1}{(1+r)} V(K_1)/K_1 - A_1/K_1 = \frac{1}{(1+r)} Q_1 - A_1/K_1$.

This is as close as I could get to the original result by Hayashi which was proved in continuous time.

5. Describe a way to test the model with a simple OLS regression if you observed K_t, I_t and market value of firms. What would happen if you didn't include taxation rules into the model, or if assumptions of part 4 did not hold?

   Let the adjustment cost function take the form $\Psi(I, K) = \frac{psi_0}{2K}(I - \delta K)^2$. The the FOC in $[I_t]$ becomes

$$(1-\tau)\psi_0(\frac{I_t}{K_t} - \delta) = q_t - (1 - \kappa - z)$$

   Rearrange to get a regression equation:

$$\frac{I_t}{K_t} = \delta + \frac{1}{\psi_0}\tilde{q}_t,$$

   where $\tilde{q}_t = \frac{q_t - (1-\kappa-z)}{(1-\tau)}$. Applying Hayashi theorem, q_t is estimated as $q_t = Q_t + A_t/K_t$, where $Q \equiv \frac{V_t}{K_t}$ is the Tobin's average q.

   If one simply used an OLS regression to estimate

$$\frac{I_t}{K_t} = \delta + \frac{1}{\psi_0}Q_t,$$

   estimates would be biased as $Q_t \neq q_t$.