Log-Linearizing Around the Steady State: A Guide with Examples

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Keywords: Log-linearization, log-deviations from the steady state, examples.

JEL categories: C60, C65

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Abstract

The paper discusses for the beginning graduate student the mathematical background and several approaches to converting nonlinear equations into log-deviations from the steady state format. Guidance is provided on when to use which approach. Pertinent examples with detailed derivation illustrate the material.

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1 Introduction

Log-linearization is a solution to the problem of reducing computational complexity for systems of numerically specified equations that need to be solved simultaneously. Such systems can be found in macroeconomics and, increasingly, also in microeconomics as numerical simulation methods are becoming more popular throughout economics. Log-linearization converts a non-linear equation into an equation that is linear in terms of the log-deviations of the associated variables from their steady state values. For small deviations from the steady state, log-deviations have a convenient economic interpretation: they are approximately equal to the percentage deviations from the steady state.

Log-linearization can greatly simplify the computational burden and, therefore, help solve a model that may otherwise be intractable. To see the degree of simplification, take as an example the equation

$$y_t = s z_t k_t^{\alpha}.$$

Log-linearization converts it into the form

$$\widetilde{y}_t = \widetilde{z}_t + \alpha \widetilde{k}_t,$$

where the log-deviations from the steady state are identified with a tilde above the variable.

This paper is motivated by the fact that log-linearization methods are not well covered in textbooks or other material for beginning graduate students. In fact, log-linearization appears to be effectively absent from all popular textbooks on mathematical methods for economists. When the material is mentioned in textbooks or discussion papers, its coverage tends to be rather cryptic (Romer 2006) or limited (Heijdra and van der Ploeg 2002, Uhlig 1995). But most importantly, it is not clear what the logic is of using one as opposed to another approach to finding log-deviations. This tends to leave beginning students confused and ill prepared to applying these methods in practice. The intent of this paper is to bring together all relevant methods of log-linearization, show their logic, contrast them, provide pertinent examples, and provide students with some guidance on when and why a particular approach works best.

The paper is organized as follows. The first section discusses some mathematical preliminaries and the substitution method of log-linearizing an equation. This is a simple method with minimal mathematical requirements. Next, this method is applied to various types of equations to illustrate (a) its universal applicability and (b) how to overcome potential stumbling blocks in practical applications. The following section discusses how log-linearizing equations can be made less tedious when it is combined with a Taylor series approximation. The concluding section points out some limitations and extensions of log-linearization.

2 The substitution method

2.1 The required tools

To understand the mathematical logic of log-linearlization requires familiarity with taking the derivative of exponential and logarithmic functions, and of Taylor series expansions.

As a reminder, the derivative of an exponential function equals the product of three items, the exponential function itself, the derivative of the exponent with respect to the decision variable, and the logarithm of the base of the exponential function.

Example 1 *For the exponential function*

$$f(x) = ce^{ax},$$

these three components are

$$ce^{ax}, a, \ln e,$$

where $\ln e = 1$. *Hence, the derivative of* f *with respect to* x *is given as*

$$f'(x) = ace^{ax}.$$

The derivative of the log function $\ln x$ is $1/x$.

Example 2 *The derivative of*

$$g(x) = a \ln bx^2$$

is given by the chain rule as

$$g'(x) = a\frac{1}{bx^2}2bx = 2\frac{a}{x}.$$

The first-order Taylor series approximation of the function h at $x = a$ is given as

$$h(x) = h(a) + h'(a)(x - a).$$

Example 3 *The function $h(x) = \ln(1+x)$ can be approximated at $x = 2$ by a first-order Taylor polynomial as*

$$h(x) \simeq \ln 3 + \frac{1}{3}(x-2) = 0.43195 + 0.3333x.$$

2.2 A basic result

Log-linearization means taking the log-deviation around a steady state value. Assume x denotes the steady state value of variable x_t. Next, define the log-deviation of variable x_t from its steady state x as

$$\widetilde{x}_t \equiv \ln x_t - \ln x. \tag{1}$$

The right hand side of equation 1 can be rewritten as

$$\ln\left(\frac{x_t}{x}\right) = \ln\left(1 + \frac{x_t - x}{x}\right).$$

The log expression can be approximated by a first-order Taylor polynomial at the steady state $x_t = x$,

$$\ln\left(1 + \frac{x_t - x}{x}\right) \simeq \ln 1 + \frac{1}{x}(x_t - x) = \frac{x_t - x}{x}.$$

The result,

$$\widetilde{x}_t \approx \frac{x_t - x}{x} = \frac{x_t}{x} - 1, \tag{2}$$

states that the log deviations of x_t from its steady state value are approximately equal to the percentage difference between x_t and its steady state value. This approximation holds for small deviations from the steady state, which highlights that log-linearization is a *local* approximation method.

Depending on what equation needs to be transformed into log-deviations format, equation 2 can be rearranged into two equivalent expressions

$$\frac{x_t}{x} \approx 1 + \widetilde{x}_t \tag{3}$$

$$x_t \approx x\left(1 + \widetilde{x}_t\right). \tag{4}$$

Equation 4 provides a means to convert equations in x_t into equations in $\widetilde{x}_t$. Such simple substitution methods work well for linear equations.

Example 4 *The national accounting identity of a closed economy without government,*

$$y_t = c_t + i_t,$$

can be converted into log-deviations form by using equation 4. Direct application yields

$$y\left(1+\widetilde{y}_t\right) = c\left(1+\widetilde{c}_t\right) + i\left(1+\widetilde{i}_t\right).$$

Typically, one wants to simplify the resulting equation. This can be done by making use of the steady state relationship that must hold for the given equation. In the present case, the steady state relationship is

$$y = c + i.$$

To make use of this relationship, multiply out the log-deviations equation,

$$y + y\widetilde{y}_t = c + c\widetilde{c}_t + i + i\widetilde{i}_t,$$

and subtract y on the left and $(c+i)$ on the right to obtain

$$y\widetilde{y}_t = c\widetilde{c}_t + i\widetilde{i}_t.$$

The final step is to divide both sides of the equation by y,

$$\widetilde{y}_t = \frac{c}{y}\widetilde{c}_t + \frac{i}{y}\widetilde{i}_t.$$

Simple substitution methods as in example 4 do not work well if the equations are more complicated, in particular if they involve variables with exponents or ratios of variables. Therefore, it would be good to have an alternative method available that is applicable for such more complicated equations.

2.3 A more general result

Start again with equation 1. But now solve the equation for x_t by taking exponents,

$$\begin{aligned} \ln x_t &= \ln x + \widetilde{x}_t \\ x_t &= e^{\ln x + \widetilde{x}_t} = e^{\ln x} e^{\widetilde{x}_t} = x e^{\widetilde{x}_t}. \end{aligned} \tag{5}$$

Dividing through by x, equation 5 can be rewritten as

$$\frac{x_t}{x} = e^{\widetilde{x}_t}.$$

Up to this point no approximation is involved. Approximating the expression $e^{\widetilde{x}_t}$ with a first-order Taylor polynomial at the point $\widetilde{x}_t = 0$ yields

$$e^{\widetilde{x}_t} \simeq 1 + e^0(\widetilde{x}_t - 0) = 1 + \widetilde{x}_t. \tag{6}$$

Applying this approximation to equation 5 leads to

$$\begin{aligned} x_t &\simeq x\,(1 + \widetilde{x}_t) \\ \frac{x_t}{x} &\approx 1 + \widetilde{x}_t, \end{aligned}$$

which happens to be identical to equations 4 and 3, respectively. At this point, one may wonder about the advantages of the derivation via exponentiation. After all, the end result is the same as the one obtained in section 2.2. The advantage will become apparent for a more complicated equation, such as the one posed in the introduction, which includes the exponential term k_t^α.

Example 5 *Converting k_t^α to log-deviations form via equation 5 yields*

$$k_t^\alpha = \left(k e^{\widetilde{k}_t}\right)^\alpha = k^\alpha e^{\alpha \widetilde{k}_t}. \tag{7}$$

Next, a first-order Taylor polynomial of the expression $e^{\alpha \widetilde{k}_t}$ at the point $\widetilde{k}_t = 0$ leads to

$$e^{\alpha \widetilde{k}_t} \simeq 1 + \alpha(\widetilde{k}_t - 0) = 1 + \alpha \widetilde{k}_t. \tag{8}$$

Substitution into equation 7 results in

$$k_t^\alpha \simeq k^\alpha(1 + \alpha \widetilde{k}_t). \tag{9}$$

The crucial point to remember from the above example is that the exponent form of the log-linearization procedure (equation 7) makes it possible to turn the exponent α into a multiplier before the Taylor approximation is employed in equation 8. This simplification is missed if the approximation of equation 4 is applied directly to the original function k_t^α. The exponent form of the log-linearization procedure also works well on ratios of variables.

Example 6 *To convert the ratio expression x_t/y_t into log-deviations form, use equation 5,*

$$\frac{x_t}{y_t} = \frac{xe^{\widetilde{x}_t}}{ye^{\widetilde{y}_t}} = \frac{x}{y}e^{\widetilde{x}_t}e^{-\widetilde{y}_t}.$$

Then apply the approximation of equation 6 to obtain

$$\frac{x}{y}e^{\widetilde{x}_t}e^{-\widetilde{y}_t} \simeq \frac{x}{y}\left(1+\widetilde{x}_t\right)\left(1-\widetilde{y}_t\right).$$

Multiplying out leads to

$$\frac{x}{y}\left(1+\widetilde{x}_t-\widetilde{y}_t-\widetilde{x}_t\widetilde{y}_t\right).$$

This condenses to

$$\frac{x}{y}\left(1+\widetilde{x}_t-\widetilde{y}_t\right)$$

because the term $\widetilde{x}_t\widetilde{y}_t$ is the product of two small numbers and, hence, negligible.

The important point of the above example is that the exponent form of the log-linearization procedure effectively eliminates the ratio before the Taylor approximation of equation 6 is employed. This simplification is missed if the approximation of equation 4 is applied directly to the given ratio x_t/y_t. The attentive reader will notice that example 6 is a corrollary of example 5 because the ratio x_t/y_t can be rewritten in the format $x_t y_t^{-1}$.[1] This highlights that the exponentiation procedure of equations 5 and 6 should always be employed if there is a variable with exponent not equal to unity in an expression that needs to be converted to log-deviations form.

3 Applications of the substitution method

3.1 Multiplicative equations

Consider the equation posed in the introduction,

$$y_t = sz_t k_t^{\alpha}. \tag{10}$$

[1]Setting $x_t = 1$ and $\alpha = -1$ replicates example 5, with variable y substituting for variable k.

To convert this equation into log-deviations format, apply the approximations from equations 4 and 7,

$$y\,(1+\widetilde{y}_t) = sz\,(1+\widetilde{z}_t)\,k^{\alpha}(1+\alpha\widetilde{k}_t). \tag{11}$$

Next, utilize the equation for the steady state to simplify equation 11. The steady state equation is given as

$$y = szk^{\alpha}. \tag{12}$$

Dividing the left hand side of equation 11 by y and the right hand side by szk^{α} generates

$$(1+\widetilde{y}_t) = (1+\widetilde{z}_t)\,(1+\alpha\widetilde{k}_t),$$

which can be solved for $\widetilde{y}_t$,

$$\widetilde{y}_t = 1+\widetilde{z}_t+\alpha\widetilde{k}_t+\alpha\widetilde{z}_t\widetilde{k}_t-1.$$

As both $\widetilde{z}_t$ and $\widetilde{k}_t$ are by assumption close to zero, its product will be negligably different from zero. Setting the product zero and simplifying yields the result

$$\widetilde{y}_t = \widetilde{z}_t+\alpha\widetilde{k}_t. \tag{13}$$

Equation 13 can be had somewhat faster by applying the definition of log-linearization directly to equation 10. This involves two steps First, take the logarithm of equation 10,

$$\ln y_t = \ln s+\ln z_t+\alpha\ln k_t.$$

Second, subtract the logarithm of the steady state of y_t (equation 12) from the left and the right sides,

$$\ln y_t-\ln y = \ln z_t-\ln z+\alpha\left(\ln k_t-\ln k\right).$$

Employing the notation of equation (1), this yields the result

$$\widetilde{y}_t = \widetilde{z}_t+\alpha\widetilde{k}_t.$$

The method of taking logs and then subtracting the log terms of the steady state equation is very convenient. However, it does not always work. It is only useful for multiplicative equations or those nearly so where taking the log removes exponents and converts multiplication into addition to significantly simplify the equation.

3.2 Nearly multiplicative equations

A nearly multiplicative equation is given by

$$x_t + a = (1-b)\frac{y_t}{z_t}.$$

Taking the log on both sides,

$$\ln(x_t + a) = \ln(1-b) + \ln y_t - \ln z_t,$$

and subtracting the log of the steady state equation,

$$\ln(x + a) = \ln(1-b) + \ln y - \ln z,$$

results in

$$\begin{aligned} \ln(x_t + a) - \ln(x + a) &= \ln y_t - \ln y - (\ln z_t - \ln z) \\ \widetilde{x_t + a} &= \widetilde{y}_t - \widetilde{z}_t, \end{aligned}$$

where the term $(1-b)$ drops out because it does not depend on time. The resulting equation contains the log-deviations of the term $x_t + a$ instead of the log-deviations of x_t. Some additional work is required to convert the former into the latter. For that purpose, employ equation 2 for both $\widetilde{x_t + a}$ and $\widetilde{x}_t$,

$$\begin{aligned} \widetilde{x_t + a} &= \frac{(x_t + a) - (x + a)}{x + a} = \frac{x_t - x}{x + a} \\ \widetilde{x}_t &= \frac{x_t - x}{x}. \end{aligned}$$

The numerators of the two equations are the same. Setting the numerator expressions equal yields

$$(x + a)\left(\widetilde{x_t + a}\right) = x\widetilde{x}_t.$$

Now solve for $\widetilde{x_t + a}$ in terms of $\widetilde{x}_t$,

$$\widetilde{x_t + a} = \frac{x}{x+a}\widetilde{x}_t.$$

Hence, the final equation in log-deviations form is given as

$$\frac{x}{x+a}\widetilde{x}_t = \widetilde{y}_t - \widetilde{z}_t.$$

3.3 Equations with expectations terms

In general, the method of taking logs and then subtracting the log terms of the steady state equation should not be used on equations that involve expectation terms, even when the equation is multiplicative. This is because taking the expectation of a log term is not the same as taking the log of an expectation term.[2] Rather, one would use equations 5 and 6. This is demonstrated with the following equation, which is in the form of a typical Euler equation that connects present and future consumption for an intertemporal utility maximization problem,

$$\frac{1}{c_t} = \beta E_t \left[\frac{(1 + r_{t+1})}{c_{t+1}} \right],$$

where E_t is an expectations operator. By equation 5, the individual components of the equation can be replaced as follows,

$$\begin{aligned} (1 + r_{t+1}) &= (1+r)\, e^{\widetilde{1+r_{t+1}}} \\ c_t &= c e^{\widetilde{c}_t} \\ c_{t+1} &= c e^{\widetilde{c}_{t+1}}. \end{aligned}$$

Substituting the above expressions gives

$$\begin{aligned} \frac{1}{c e^{\widetilde{c}_t}} &= \beta E_t \left[\frac{(1+r)\, e^{\widetilde{1+r_{t+1}}}}{c e^{\widetilde{c}_{t+1}}} \right] \\ \frac{1}{e^{\widetilde{c}_t}} &= \beta\, (1+r)\, E_t \left(\frac{e^{\widetilde{1+r_{t+1}}}}{e^{\widetilde{c}_{t+1}}} \right) \end{aligned}$$

[2] This is the result of Jensen's inequality, which implies $\ln(Ex) \geq E \ln x$ for the log function. Only for a linear function $f(x)$ is $f(Ex) = Ef(x)$.

$$\begin{aligned} e^{-\widetilde{c}_t} &= \beta(1+r)\,E_t e^{\widetilde{1+r}_{t+1}} e^{-\widetilde{c}_{t+1}} \\ 1 &= \beta(1+r)\,E_t e^{\widetilde{1+r}_{t+1}} e^{-\widetilde{c}_{t+1}} e^{\widetilde{c}_t}. \end{aligned}$$

So far, no approximation has been applied. Employ now the approximation of equation 6 to each one of the exponential terms,

$$1 = \beta(1+r)\,E_t\left[\left(1+\widetilde{1+r}_{t+1}\right)(1-\widetilde{c}_{t+1})(1+\widetilde{c}_t)\right].$$

The bracket term on the right hand side needs to be multiplied out,

$$1 = \beta(1+r)\,E_t\left(\begin{array}{c} 1+\widetilde{c}_t-\widetilde{c}_{t+1}+\widetilde{1+r}_{t+1}-\widetilde{c}_t\widetilde{c}_{t+1}-\widetilde{1+r}_{t+1}\widetilde{c}_{t+1} \\ +\widetilde{1+r}_{t+1}\widetilde{c}_t-\widetilde{1+r}_{t+1}\widetilde{c}_t\widetilde{c}_{t+1} \end{array}\right).$$

The last four terms in parenthesis are products of log-deviations from steady state and, therefore, very small. Setting them zero and removing the number one from the parenthesis term yields

$$1 = \beta(1+r) + \beta(1+r)\,E_t\left(\widetilde{c}_t-\widetilde{c}_{t+1}+\widetilde{1+r}_{t+1}\right).$$

Economic theory tells us that in steady state $\beta = 1/(1+r)$. Making use of this steady state condition simplifies the equation to

$$0 = E_t\left(\widetilde{c}_t-\widetilde{c}_{t+1}+\widetilde{1+r}_{t+1}\right).$$

The final step is the conversion of the term $\widetilde{1+r}_{t+1}$ into a term involving $\widetilde{r}_t$. Following the example in the last section, we employ the approximation from equation 2 to obtain,

$$\begin{aligned} \widetilde{1+r}_{t+1} &\approx \frac{1+r_{t+1}-(1+r)}{1+r} = \frac{r_{t+1}-r}{1+r} \\ \widetilde{r}_{t+1} &\approx \frac{r_{t+1}-r}{r}. \end{aligned}$$

Solving both equations for the numerator terms and setting them equal yields

$$(1+r)\left(\widetilde{1+r}_{t+1}\right) = r\widetilde{r}_{t+1}$$

or, when solved for $\widetilde{1+r_{t+1}}$,

$$\widetilde{1+r_{t+1}} = \left(\frac{r}{1+r}\right)\widetilde{r}_{t+1}.$$

Substituting the result gives the final equation in log-deviations form,

$$0 = E_t\left[\widetilde{c}_t - \widetilde{c}_{t+1} + \left(\frac{r}{1+r}\right)\widetilde{r}_{t+1}\right].$$

3.4 Equations in logs

Macroeconomic models often contain log equations for stochastic technology shocks of the type

$$\ln z_t = z_0 + \rho \ln z_{t-1} + \epsilon_t,$$

where ϵ_t is a disturbance term. To convert to log-deviations format, replace the time subscripted variables per equation 5,

$$\begin{aligned} \ln z e^{\widetilde{z}_t} &= z_0 + \rho \ln z e^{\widetilde{z}_{t-1}} + \epsilon_t \\ \ln z + \widetilde{z}_t &= z_0 + \rho\left(\ln z + \widetilde{z}_{t-1}\right) + \epsilon_t. \end{aligned}$$

Using knowledge about the steady state can simplify the above equation. In particular, in steady state the following obtains,

$$\ln z = z_0 + \rho \ln z.$$

By subtracting $\ln z$ on the left and $(z_0 + \rho \ln z)$ on the right, the log equation simplifies to

$$\widetilde{z}_t = \rho \widetilde{z}_{t-1} + \epsilon_t.$$

4 Log-linearizing via Taylor series approximation

So far, only simple algebraic substitutions have been used to derive equations in log-deviations format. No more is required for any equation. However, the substitution method via equations 5 and 6 may become rather time consuming to use for more complicated equations. Significant time savings can typically be obtained by first using a Taylor series approximation before applying the definitions of log-deviations.

4.1 Univariate case

To see the logic of this method and its potential for time savings, consider the equation

$$x_{t+1} = f(x_t),$$

where f is a possibly complicated nonlinear function. A first-order Taylor polynomial of this equation at the steady state $x_t = x$ gives

$$x_{t+1} \approx f(x) + f'(x)(x_t - x).$$

As $x = f(x)$ in steady state, the equation can be rewritten as

$$x_{t+1} \approx x + f'(x)(x_t - x).$$

Dividing by x,

$$\frac{x_{t+1}}{x} \approx \frac{x}{x} + f'(x)\frac{(x_t - x)}{x},$$

and employing equation 3 on the left and equation 2 on the right yields

$$\begin{aligned} 1 + \widetilde{x}_{t+1} &= 1 + f'(x)\widetilde{x}_t \\ \widetilde{x}_{t+1} &= f'(x)\widetilde{x}_t. \end{aligned} \tag{14}$$

Hence, log-linearization involves no more than taking the first derivative of the function $f(x_t)$. To see this methodology in action, consider the following example.

Example 7 *Assume an equation similar to the example given in the introduction,*

$$k_{t+1} = sk_t^{\alpha} + (1-\delta)k_t. \tag{15}$$

As a first step in the conversion to log-deviations, a first-order Taylor series expansion at the steady state $k_t = k$ *results in*

$$k_{t+1} \approx [sk^{\alpha} + (1-\delta)k] + \left[\alpha sk^{\alpha-1} + (1-\delta)\right](k_t - k).$$

In steady state, the equation $k = sk^{\alpha} + (1-\delta)k$ *holds. Therefore,*

$$k_{t+1} \approx k + \left[\alpha sk^{\alpha-1} + (1-\delta)\right](k_t - k).$$

Now divide by k,

$$\frac{k_{t+1}}{k} \approx 1 + \left[\alpha s k^{\alpha-1} + (1-\delta)\right] \frac{(k_t - k)}{k},$$

and use equations 3 and 2,

$$\widetilde{k}_{t+1} \approx \left[\alpha s k^{\alpha-1} + (1-\delta)\right] \widetilde{k}_t. \tag{16}$$

Equation 16 is the log-linearized version of equation 15.[3] *As usual, further simplifications of equation 16 are possible by employing the steady state relationship,*

$$k = sk^{\alpha} + (1-\delta)k.$$

In this case, it is convenient to solve the steady state equation for $sk^{\alpha-1}$,

$$\begin{aligned} 1 &= sk^{\alpha-1} + (1-\delta) \\ sk^{\alpha-1} &= 1 - (1-\delta). \end{aligned}$$

Replacing the term $sk^{\alpha-1}$simplifies equation 16 to

$$\widetilde{k}_{t+1} \approx \left[1 - (1-\alpha)\delta\right] \widetilde{k}_t.$$

4.2 Multivariate case

First-order Taylor approximations can also be used to convert equations with more than one endogenous variable to log-deviations form. The result for two variables simply follows the steps for the one-variable case in section 4.1. In particular, start with an equation like

$$x_{t+1} = g(x_t, y_t),$$

and employ a first-order Taylor approximation at the steady state values $x_t = x$ and $y_t = y$,

$$x_{t+1} \approx g(x,y) + g'_x(x,y)(x_t - x) + g'_y(x,y)(y_t - y). \tag{17}$$

[3] Note that equation 16 can be obtained from equation 15 in one step simply be employing equation 14. After all, the bracket term in equation 16 is nothing but the derivative of equation 15 with respect to k_t, evaluated at $k_t = k$.

In steady state,

$$x = g(x, y),$$

which can be used to rewrite equation 16 as

$$x_{t+1} \approx x + g'_x(x, y)(x_t - x) + g'_y(x, y)(y_t - y).$$

Dividing through by x and multiplying and dividing the last term on the right by y yields

$$\frac{x_{t+1}}{x} \approx 1 + g'_x(x, y)\frac{(x_t - x)}{x} + g'_y(x, y)\frac{y}{x}\frac{(y_t - y)}{y}.$$

Sequentially employing equations 3 and 2 and rearranging terms generates the following sequence of equations,

$$\begin{aligned} 1 + \widetilde{x}_{t+1} &\approx 1 + g'_x(x, y)\widetilde{x}_t + g'_y(x, y)\frac{y}{x}\widetilde{y}_t \\ \widetilde{x}_{t+1} &\approx g'_x(x, y)\widetilde{x}_t + g'_y(x, y)\frac{y}{x}\widetilde{y}_t \\ x\widetilde{x}_{t+1} &\approx g'_x(x, y)x\widetilde{x}_t + g'_y(x, y)y\widetilde{y}_t. \end{aligned} \tag{18}$$

Example 8 *A two-variable version of example 7 is given as*

$$k_{t+1} = sz_t k_t^\alpha + (1 - \delta)k_t. \tag{19}$$

The equation contains the variables k_t and z_t on the right hand side. Employing equation 18 directly to equation 19 and simplifying the result generates

$$\begin{aligned} k\widetilde{k}_{t+1} &\approx \left[\alpha szk^{\alpha-1} + (1 - \delta)\right] k\widetilde{k}_t + (sk^\alpha)\, z\widetilde{z}_t \\ \widetilde{k}_{t+1} &\approx \left[\alpha szk^{\alpha-1} + (1 - \delta)\right] \widetilde{k}_t + \left(szk^{\alpha-1}\right) \widetilde{z}_t. \end{aligned} \tag{20}$$

Again, the steady state relationship,

$$\begin{aligned} k &= szk^\alpha + (1 - \delta)k \\ 1 &= szk^{\alpha-1} + (1 - \delta), \end{aligned}$$

can be used to further simplify equation 20,

$$\widetilde{k}_{t+1} \approx [1 - (1 - \alpha)\delta]\, \widetilde{k}_t + \delta\widetilde{z}_t.$$

It is apparent that the use of the Taylor polynomial can significantly reduce the time needed to generate the log-deviations form of more complicated equations. This applies, of course, all the more if one makes use of equations 14 and 18 directly, rather than deriving them from scratch on every new equation that needs to be log-linearized.

5 Conclusion

This paper has introduced the basic principles of log-linearizing equations. The main purpose has been (a) to bring together in one place the methodology that can be found scattered across various sources, (b) to illustrate with one consistent notation how the main approaches relate to each other, and (c) to provide some guidelines, in conjunction with a set of pertinent examples, of how the various methods are applied in practice.

The discussion in this paper has been limited to difference equations because difference equations are currently much more popular in applied economics than differential equations. It should be noted, however, that the methods covered here can be applied in only slightly modified form also to differential equations.[4]

Finally, the reader needs to be cautioned that log-linearization, although a convenient tool, is not an economically sensible simplification for all models. For example, if the variability of a random variable is important, such as in the modeling of risk, log-linearization is not appropriate because only the mean of the random variable is considered by log-linearized equations not its variance. Other methods of making equations computationally tractable need to be employed in such cases. Perturbation methods and other techniques that make use of higher order terms have gained popularity for this reason (Judd 1998, Miranda and Fackler 2002). This is where the interested reader may want to turn next.

References

Heijdra, Ben J. and Frederick van der Ploeg. 2002. *Foundations of Modern Macroeconomics.* Oxford University Press, Oxford and New York.

Judd, Kenneth L. 1998. *Numerical Methods in Economics*, MIT Press, Cambridge, Massachussets.

Miranda, Mario J. and Paul L. Fackler. 2002. *Applied Computational Economics and Finance.* MIT Press, Cambridge and London.

[4] See, for example, Heijdra and van der Ploeg (2002, chapter 15).

Romer, David. 2006. *Advanced Macroeconomics*. 3rd edition, McGraw-Hill/Irwin, New York.

Uhlig, Harald. 1995. A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily. Discussion Paper, Institute for Empirical Macroeconomics 101, Federal Reserve Bank of Minneapolis, Minneapolis.