Econ 714: Problem Set 1 - Solution[1]

1

(a) From lecture notes, we have:

$$w_R - z = \frac{\beta p}{1-\beta(1-s)} \int_{w_R}^{\infty} (w' - w_R) dF(w')$$

Add and subtract $\frac{\beta p}{1-\beta(1-s)} \left(\int_0^{w_R} w' dF(w') + w_R \int_0^{w_R} dF(w') \right)$ on the right hand-side. After collecting terms, we get:

$$\begin{aligned} w_R - z &= \frac{\beta p}{1-\beta(1-s)} \left(\int_0^{\infty} w' dF(w') - w_R \int_0^{\infty} dF(w') - \int_0^{w_R} (w' - w_R) dF(w') \right) \\ &= \frac{\beta p}{1-\beta(1-s)} \left(\mathbb{E}w - w_R - \int_0^{w_R} (w' - w_R) dF(w') \right) \end{aligned}$$

where $\mathbb{E}w$ is the expectation of wage under the distribution $F(w')$. Multiply by $(1-\beta(1-s))$ both sides and rearrange terms. Finally, by integrating the last integral by parts, we get:

$$(1-\beta(1-s)+\beta p) w_R - (1-\beta(1-s)) z = \beta p E w + \beta p \int_0^{w_R} F(w') dw' \quad (1)$$

(b) If G be a mean preserving spread of F, then $\int_0^b G(w')dw' \geq \int_0^b F(w')dw'$. Let $h_f(w) = \int_0^w F(w')dw'$ and $h_g(w) = \int_0^w G(w')dw'$. Then, for any w, $h_g(w) \geq h_f(w)$ and so reservation wage is (weakly) higher $w_{R,g} \geq w_{R,f}$.[2]

(c) From the lecture notes, we have:

$$w_R - z = \frac{\beta p}{1-\beta(1-s)} \int_{w_R}^{\infty} (w' - w_R) dF(w)$$

The term $\int_{w_R}^{\infty} (w' - w_R) dF(w)$ is decreasing in w. Then, if we re-write this as:

$$\frac{w_R - z}{\int_{w_R}^{\infty} (w' - w_R) dF(w)} = \frac{\beta p}{1-\beta(1-s)}$$

[1] By Anton Babkin. February 15, 2016.

[2] Let $h(s) = \int_0^s F(p)dp$, then $h'(s) = F(s) > 0$ and $h''(s) = f(s) > 0$ so the function $h(s)$ is convex in s. See Ljungqvist and Sargent textbook for more detail.

Then, the left-hand side will be increasing in w_R. If p falls, the reservation wage falls as well.

Steady-state unemployment rate is determined by:

$$\begin{aligned} up(1-F(w_R)) &= s(1-u) \\ u &= \frac{s}{p(1-F(w_R))+s} \end{aligned} \tag{2}$$

We cannot really say what will happen to steady-state unemployment rate if p falls without imposing some conditions on $F(w)$.

2

First notice that at the highest level productivity $x = 1$ there is no incentive to search for a new job, because it won't yield a higher wage. So when agent gets a new job, his value is $W^n(1)$.

Value of the unemployed is almost like in a standard Mortensen-Pissarides model:

$$rU = z + f(W^n(1) - U)$$

Employed worker will not switch his search decision unless a productivity shock arrives. Once the shock hits, the job is either destroyed or continues with a new level of productivity - this is when worker can decide to switch between searching and not.

Value of the non-searching employed:

$$rW^n(x) = w(x) + \lambda \left[\int_0^R U dG(x') + \int_R^1 \max\{W^n(x'), W^s(x')\} dG(x') - W^n(x) \right]$$

(c.)

Value of the searching employed:

$$\begin{aligned} rW^s(x) =& w(x) - \sigma + f\left[W^n(1) - W^s(x)\right] \\ &+ \lambda \left[\int_0^R U dG(x') + \int_R^1 \max\{W^n(x'), W^s(x')\} dG(x') - W^s(x) \right] \end{aligned}$$

3

(a) The social planner's problem is

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\lambda u(c_t^1) + (1-\lambda) u(c_t^2)\right]$$
$$\text{s.t. } c_t^1 + c_t^2 = e_t^1 + e_t^2 \quad \forall t$$

where we know that $e_t^1 = 1$ and $e_t^2 = 0$ for $0 \le t < 21$, and $e_t^1 = 0$ and $e_t^2 = 1$ for $t \ge 21$. Hence the resource constraint can be rewritten as

$$c_t^1 + c_t^2 = 1 \tag{3}$$

Attaching the Lagrangian multipliers $\{\mu_t\}$ to the constraints, the FOC's are

$$c_t^1 : \lambda u'(c_t^1) = \mu_t$$
$$c_t^2 : (1-\lambda) u'(c_t^2) = \mu_t$$

Equating the above we have

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{1-\lambda}{\lambda} \tag{4}$$

and using the resource constraint we have

$$\frac{u'(c_t^1)}{u'(1-c_t^1)} = \frac{1-\lambda}{\lambda} \tag{5}$$

Notice that from (3) we have, for $t \neq t'$, that if $c_t^1 \ge c_{t'}^2$, then it must be that $c_t^2 < c_{t'}^2$. But then $\frac{u'(c_t^1)}{u'(c_t^2)} < \frac{u'(c_{t'}^1)}{u'(c_{t'}^2)}$, which vioates (4), so it must be that

$$c_t^1 = c^1$$
$$c_t^2 = c^2$$

The value of c_t^1 (and hence of c^2) is given by (5) and depends on λ and the utility function.

(b) A competitive equilibrium is an allocation $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and a set of prices $\{p_t\}_{t=0}^{\infty}$ such that:

- Agent $i \in \{1, 2\}$ maximizes utility:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i)$$
$$\text{s.t.} \sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t e_t^i \tag{6}$$

- Markets clear:

$$c_t^1 + c_t^2 = 1 \tag{7}$$

(c) Attaching the Lagrangian multiplier κ_i to agent i's budget constraint, the FOC of this problem is

$$c_t^i : \beta^t u'(c_t^i) = \kappa^i p_t \tag{8}$$

Normalize by setting $p_0 = 1$, so

$$\kappa^i = u'(c_0^i)$$

Combining the FOC's for the two agents we have

$$\frac{u'(c_t^1)}{u'(c_t^2)} = \frac{u'(c_0^1)}{u'(c_0^2)} \equiv \eta \tag{9}$$

Notice that from (7) we have that if $c_t^1 \geq c_{t'}^2$, then it must be that $c_t^2 < c_{t'}^2$. But then $\frac{u'(c_t^1)}{u'(c_t^2)} < \frac{u'(c_{t'}^1)}{u'(c_{t'}^2)}$, which vioates (9), so it must be that

$$\begin{aligned} c_t^1 &= c^1 \\ c_t^2 &= c^2 \end{aligned} \tag{10}$$

Using this in (8) for agent i for periods 0 and t we have

$$p_t = \beta^t \tag{11}$$

Using (10) and (11) in (6) we have

$$\sum_{t=0}^{\infty} \beta^t c^1 = \sum_{t=0}^{20} \beta^t$$

which implies $c^1 = 1 - \beta^{21}$, and using (7) we have $c^2 = \beta^{21}$.

Thus, the competitive equilibrium is $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and $\{p_t\}_{t=0}^{\infty}$, where

$$\begin{aligned} c_t^1 &= 1 - \beta^{21} \quad \forall t \\ c_t^2 &= \beta^{21} \quad \forall t \\ p_t &= \beta^t \quad \forall t \end{aligned}$$

Clearly, this competitive equilibrium is Pareto optimal for the appropriate choice of λ (1st welfare theorem), while the Pareto optimum is a competitive equilibrium with transfers (2nd welfare theorem).

(d) The price of the claim to consumer 1's endowment process must be equal to the price of purchasing an equivalent sequence of consumption $c_t^1 = 1$ for $t = 0, 1, ..., 20$ and $c_t^1 = 0$ for $t > 20$. The same is true for consumer 2's endowment process and for the aggregate endowment process. Thus:

$$\begin{aligned} p_e^1 &= \sum_{t=0}^{20} p_t = \sum_{t=0}^{20} \beta^t = \frac{1-\beta^{21}}{1-\beta} \\ p_e^2 &= \sum_{t=21}^{\infty} p_t = \sum_{t=21}^{\infty} \beta^t = \frac{\beta^{21}}{1-\beta} \\ p_e^1 &= p_e^1 + p_e^2 = \sum_{t=1}^{\infty} p_t = \frac{1}{1-\beta} \end{aligned}$$

4

(a) Household problem, $i = 1, 2$:

$$\max_{\{c_t^i(a^t,s^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{(a^t,s^t)} \beta^t \log(c_t^i(a^t, s^t)) Pr(a^t, s^t)$$

$$\text{s.t.} \sum_{t=0}^{\infty} \sum_{(a^t,s^t)} q_t^0(a^t, s^t) c_t^i(a^t, s^t) = \sum_{t=0}^{\infty} \sum_{(a^t,s^t)} q_t^0(a^t, s^t) e_t^i(a_t, s_t)$$
$$c_t^i(a^t, s^t) \geq 0$$
$$e_t^1(a_t, s_t) = a_t + s_t$$
$$e_t^2(a^t, s^t) = a_t + 1 - s_t$$

(b) The Arrow-Debreu competitive equilibrium is the sequence of allocations $\{\{c_t^i(a^t, s^t)\}_{t=0}^{\infty}\}_{i=1}^{2}$ and prices $\{q_t^0(a^t, s^t)\}_{t=0}^{\infty}$ that solves household problem and satisfies market clearing condition:

$$c_t^1(a^t, s^t) + c_t^2(a^t, s^t) = e_t^1(a_t, s_t) + e_t^2(a_t, s_t)$$

(c) Note that with the utility function being $\log(c)$, the Inada conditions are satisfied, the solution is interior and budget constraint holds with equality, and we can drop the nonnegativity constraint. By Negishi algorithm, consider the social planner's problem with the Pareto weights (w^1, w^2) and $w^1 + w^2 = 1$:

$$\max_{\{\{c_t^i(a^t,s^t)\}_{t=0}^{\infty}\}_{i=1}^{2}} \sum_{i=1}^{2} w^i \sum_{t=0}^{\infty} \sum_{(a^t,s^t)} \beta^t \log(c_t^i(a^t, s^t)) Pr(a^t, s^t)$$
$$\text{s.t. } c_t^1(a^t, s^t) + c_t^2(a^t, s^t) = e_t^1(a_t, s_t) + e_t^2(a_t, s_t)$$

Attaching the Lagrangian multiplier λ_t to the budget constraint, FOC w.r.t. $c_t^i(a^t, s^t)$ is:

$$\frac{w^i \beta^t Pr(a^t, s^t)}{c_t^i(a^t, s^t)} = \lambda_t$$

Divide two conditions for $i = 1, 2$:

$$\frac{w^1}{c_t^1(a^t, s^t)} = \frac{w^2}{c_t^2(a^t, s^t)}$$

Use this together with $e_t(a_t, s_t) = e_t^1(a_t, s_t) + e_t^2(a_t, s_t) = 2a_t + 1$ and the feasibility constraint to obtain optimal consumption allocations $c_t^i(a^t, s^t) = w^i e_t(a_t, s_t)$.

Now turn back to household problem. Attaching the Lagrangian multiplier μ^i to the budget constraint of household i, obtain FOC:

$$\frac{\beta^t Pr(a^t, s^t)}{c_t^i(a^t, s^t)} = \mu^i q_t^0(a^t, s^t)$$

Normalizing the price at date $t = 0$ and state $(a_0, s_0) = (0, 0)$: $q_0^0(0, 0) = \overline{\pi}_a \overline{\pi}_b$, evaluate the above expression at state $(a_0, s_0) = (0, 0)$, and note that $e_0(0, 0) = 1$, we can solve for the Lagrangian multiplier:

$$\mu^i = \frac{1}{c_0^i(0, 0)} = \frac{1}{w^i e_0(0, 0)} = \frac{1}{w^i}$$

Substitute into the first-order condition to obtain the AD securities price as:

$$q_t^0(a^t, s^t) = \frac{\beta^t Pr(a^t, s^t)}{\mu^i c_t^i(a^t, s^t)} = \frac{w^i \beta^t Pr(a^t, s^t)}{w^i e_t(a_t, s_t)} = \frac{\beta^t Pr(a^t, s^t)}{2a_t + 1}$$

Withe price and optimal consumption allocation, substitute into budget constraint to solve for pareto weight for individual $i = 1$ and that $w^2 = 1 - w^1$:

$$\begin{aligned}
\sum_{t=0}^{\infty} \sum_{(a^t, s^t)} \frac{\beta^t Pr(a^t, s^t)}{e_t(a_t, s_t)} w^1 e_t(a_t, s_t) &= \sum_{t=0}^{\infty} \sum_{(a^t, s^t)} \frac{\beta^t Pr(a^t, s^t)}{e_t(a_t, s_t)} e_t^i(a_t, s_t) \\
w^1 \sum_{t=0}^{\infty} \beta^t \sum_{(a^t, s^t)} Pr(a^t, s^t) &= \sum_{t=0}^{\infty} \beta^t \sum_{(a^t, s^t)} Pr(a^t, s^t) \frac{a_t + s_t}{2a_t + 1} \\
w^1 \sum_{t=0}^{\infty} \beta^t &= \sum_{t=0}^{\infty} \beta^t \sum_{(a_t, s_t)} Pr(a_t, s_t) \frac{a_t + s_t}{2a_t + 1} \\
w^1 \sum_{t=0}^{\infty} \beta^t &= \left(\frac{1}{3}\overline{\pi}_a - \frac{1}{3}\overline{\pi}_s - \frac{2}{3}\overline{\pi}_a \overline{\pi}_s + \frac{2}{3} \right) \sum_{t=0}^{\infty} \beta^t \\
w^1 &= \frac{1}{3}\overline{\pi}_a - \frac{1}{3}\overline{\pi}_s - \frac{2}{3}\overline{\pi}_a \overline{\pi}_s + \frac{2}{3} \\
w^2 &= \frac{1}{3}\overline{\pi}_a - \frac{1}{3}\overline{\pi}_s - \frac{2}{3}\overline{\pi}_a \overline{\pi}_s - \frac{1}{3}
\end{aligned}$$

To understand how $\sum_{(a^t, s^t)} Pr(a^t, s^t) \frac{a_t + s_t}{2a_t + 1}$ turns into $\sum_{(a_t, s_t)} Pr(a_t, s_t) \frac{a_t + s_t}{2a_t + 1}$ in the second equation, consider a simpler case with a single two-state Markov random variable $z_t \in \{0, 1\}$. Let stationary distribution be $(\bar{p}_0, \bar{p}_1)$, and transition probability from state i to state j be p_{ij}.

For $t = 0$, $Pr(z_0 = 0) = \bar{p}_0$ and $Pr(z_0 = 1) = \bar{p}_1$. At $t = 1$, probability of history $Pr(z^t) = Pr(z_0, z_1) = Pr(z_0)Pr(z_1|z_0)$. For example, $Pr(0, 0) = \bar{p}_0 p_{00}$. Now, group summation terms over histories at $t = 1$:

$$[Pr(0, 0) + Pr(1, 0)] + [Pr(0, 1) + Pr(1, 1)]$$
$$[\bar{p}_0 p_{00} + \bar{p}_1 p_{10}] + [\bar{p}_0 p_{01} + \bar{p}_1 p_{11}]$$

But by the definition of stationary distribution, $\bar{p}_0 p_{00} + \bar{p}_1 p_{10} = \bar{p}_0$ and $\bar{p}_0 p_{01} + \bar{p}_1 p_{11} = \bar{p}_1$, so the sum simplifies to $\bar{p}_0 + \bar{p}_1$. Same reasoning applies $\forall t$.

Now, using the values for w^i, can write down AD equilibrium:

$$
\begin{aligned}
c_t^1(a^t, s^t) &= \left(\frac{1}{3}\overline{\pi}_a - \frac{1}{3}\overline{\pi}_s - \frac{2}{3}\overline{\pi}_a\overline{\pi}_s + \frac{2}{3}\right)(2a_t + 1) \\
c_t^2(a^t, s^t) &= \left(\frac{1}{3}\overline{\pi}_a - \frac{1}{3}\overline{\pi}_s - \frac{2}{3}\overline{\pi}_a\overline{\pi}_s - \frac{1}{3}\right)(2a_t + 1) \\
q_t^0(a^t, s^t) &= \frac{\beta^t Pr(a^t, s^t)}{2a_t + 1}
\end{aligned}
$$

As always under complete markets, consumption does not depend on idiosyncratic risk s_t. Consumers get a constant fraction of aggregate endowment which only varies with a_t.

(d) Price of one-period contingent claims (Arrow securities) can be found as

$$
q_{t+1}^t(a^{t+1}, s^{t+1}) = \frac{q_{t+1}^0(a^{t+1}, s^{t+1})}{q_t^0(a^t, s^t)} = \frac{\frac{\beta^{t+1} Pr(a^{t+1}, s^{t+1})}{2a_{t+1}+1}}{\frac{\beta^t Pr(a^t, s^t)}{2a_t+1}}
$$

$$
= \beta \frac{2a_t + 1}{2a_{t+1} + 1} Pr(a_{t+1}, s_{t+1} | a_t, s_t) = \beta \frac{2a_t + 1}{2a_{t+1} + 1} Pr(a_{t+1}|a_t) Pr(s_{t+1}|s_t)
$$

There are 16 prices in total, for 4 possible states at $t + 1$ after each of 4 possible states at t. With given parameters, prices are:

$$
\begin{aligned}
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 0, s_t = 0) = 0.684 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 0, s_t = 1) = 0.152 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 1, s_t = 0) = 0.7695 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 1, s_t = 1) = 0.171 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 1|a_t = 0, s_t = 0) = 0.076 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 1|a_t = 0, s_t = 1) = 0.608 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 1|a_t = 1, s_t = 0) = 0.0855 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 1|a_t = 1, s_t = 1) = 0.681 \\
q_{t+1}^{t}(a_{t+1} = 1, s_{t+1} = 0|a_t = 0, s_t = 0) = 0.057 \\
q_{t+1}^{t}(a_{t+1} = 1, s_{t+1} = 0|a_t = 0, s_t = 1) = 0.0127 \\
q_{t+1}^{t}(a_{t+1} = 1, s_{t+1} = 0|a_t = 1, s_t = 0) = 0.5985 \\
q_{t+1}^{t}(a_{t+1} = 1, s_{t+1} = 0|a_t = 1, s_t = 1) = 0.133 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 0, s_t = 0) = 0.0063 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 0, s_t = 1) = 0.0507 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 1, s_t = 0) = 0.0665 \\
q_{t+1}^{t}(a_{t+1} = 0, s_{t+1} = 0|a_t = 1, s_t = 1) = 0.532
\end{aligned}
$$

(e) The one-period ahead riskless claim to one unit of consumption depends on the current state of the world, consumer must buy one unit of Arrow securities contingent for each possible state of the world in the next period. Hence, we have the following prices:

- If $(a_t, s_t) = (0, 0)$, the price is $0.684 + 0.076 + 0.057 + 0.0063 = 0.8233$
- If $(a_t, s_t) = (0, 1)$, the price is $0.152 + 0.608 + 0.0127 + 0.0507 = 0.8233$
- If $(a_t, s_t) = (1, 0)$, the price is $0.7695 + 0.0855 + 0.5985 + 0.0665 = 1.52$
- If $(a_t, s_t) = (1, 1)$, the price is $0.171 + 0.684 + 0.133 + 0.532 = 1.52$