Econ 714: Problem Set 2 - Solution[1]

1

(a) A competitive equilibrium under autarky is, for each country $i = a, b$, a sequence of prices $\{p^i_{1t}, p^i_{st}\}_{t=0}^{\infty}$ for good 1 and a claim to a tree, a sequence of returns $\{r^i_t\}_{t=0}^{\infty}$ for the risk-free bond, a goods allocation $\{c^i_{1t}, c^i_{2t}\}_{t=0}^{\infty}$ and an asset allocation $\{b^i_t, s^i_t\}$ of bonds and shares in trees such that:
(i) The household allocation solves the household problem of maximizing utility subject to the flow budget constraint (suppressing superscripts):

$$p_{1t}c_{1t} + c_{2t} + p_{st}s_{t+1} + \frac{1}{1+r_t}b_{t+1} \leq b_t + (p_{st} + p_{1t}e_{1t} + 1)\, s_t \tag{1}$$

(ii) The goods market clears (for all t, $i = a, b$):

$$c^i_{1t} = e^i_{1t}, \quad c^i_{2t} = 1. \tag{2}$$

(iii) The asset market clears (for all t, $i = a, b$):

$$b^i_t = 0, \quad s^i_t = 1.$$

(b) Since the problem is Markov, we will find a Markov equilibrium and represent the optimization problem via dynamic programming, and hence drop time subscripts. It is easiest if we define the state variable as a representative agent's wealth w, where:

$$w = b + (p_s + p_1 e_1 + 1)\, s.$$

Then the budget constraint is:

$$p_1 c_1 + c_2 + p_s s' + \frac{1}{1+r}b' \leq w. \tag{3}$$

Then the Bellman equation is:

$$V(w) = \max_{c_1, c_2, s', b'} \left\{ \frac{\left(c_1^{\alpha} c_2^{1-\alpha}\right)^{1-\gamma}}{1-\gamma} + \beta E\left(V(w')|w\right) \right\} \tag{4}$$

subject to (3) and

$$w' = b' + (p'_s + p'_1 e'_1 + 1)\, s'. \tag{5}$$

The algebra is easiest if we do not impose the constraint (3) but instead let λ be the Lagrange multiplier on it. Then the first order and envelope

[1] By Anton Babkin. March 1, 2016. Original solutions for problems 1-3 by Tim Lee.

conditions are:

$$\begin{aligned}\left(c_1^{\alpha} c_2^{1-\alpha}\right)^{-\gamma} \alpha c_1^{\alpha-1} {c_2}^{1-\alpha} &= p_1 \lambda & (6)\\ \left(c_1^{\alpha} c_2^{1-\alpha}\right)^{-\gamma} (1-\alpha) c_1^{\alpha} c_2^{-\alpha} &= \lambda & (7)\\ p_s \lambda &= \beta E\left(V'(w')(p_s' + p_1' e_1' + 1)|w\right) & (8)\\ \frac{1}{1+r}\lambda &= \beta E\left(V'(w')|w\right) & (9)\\ V'(w) &= \lambda & (10)\end{aligned}$$

For later use, note that if we divide (6) by (7) we get an expression for p_1:

$$p_1 = \frac{\alpha}{1-\alpha}\frac{c_2}{c_1}. \tag{11}$$

Similarly, using (8), (9), and (10) we get Euler equations for the share price and the risk free rate:

$$\begin{aligned} p_s &= \beta E\left(\frac{\lambda'}{\lambda}(p_s' + p_1' e_1' + 1)\Big| w\right) & (12)\\ \frac{1}{1+r} &= \beta E\left(\frac{\lambda'}{\lambda}\Big| w\right) & (13)\end{aligned}$$

(c) Using our expressions above, these follow by imposing the autarkic equilibrium conditions. Using this in (7), (11)-(13) become:

$$\begin{aligned} p_1 &= \frac{\alpha}{1-\alpha}\frac{1}{e_1}\\ p_s &= \beta E\left(\left(\frac{e_1'}{e_1}\right)^{\alpha(1-\gamma)} (p_s' + p_1' e_1' + 1)\Big| w\right)\\ \frac{1}{1+r_A} &= \beta E\left(\left(\frac{e_1'}{e_1}\right)^{\alpha(1-\gamma)} \Big| w\right)\end{aligned}$$

We can be a little more explicit in the expression for r_A by using the structure of the endowments. In particular, in country a we get:

$$\begin{aligned} \frac{1}{1+r_A^a} &= \beta\left(p + (1-p)\left(\frac{1-e_h}{e_h}\right)^{\alpha(1-\gamma)}\right) \quad \text{if } \ e_1^a = e_h\\ &= \beta\left(p\left(\frac{e_h}{1-e_h}\right)^{\alpha(1-\gamma)} + (1-p)\right) \quad \text{if } \ e_1^a = 1-e_h\end{aligned}$$

Similar expressions hold for country b, and the interest rates will vary across countries and across states.

(d) Under free trade, the relevant goods market clearing conditions become:

$$c_1^a + c_1^b \leq 1, \ c_2^a + c_2^b \leq 2$$

Notice in particular that there is no aggregate uncertainty and since we allow for asset trade as well there are complete markets. Therefore (as we showed in class) the equilibrium allocation will consist of constant consumption. (The consumption shares are determined by the initial endowments, but we don't need them here.) Therefore from (7) we get $\lambda'/\lambda = 1$ and so (13) implies:

$$\frac{1}{1+r_f^i} = \beta.$$

This is clearly the same across countries, which must be the case by arbitrage.

(e) Since goods are traded freely across countries, this means that by arbitrage we must have goods prices equalized at $p_1^a = p_1^b = p_1$. The relevant goods market clearing conditions are as in the previous part:

$$c_1^a + c_1^b \leq 1,\ c_2^a + c_2^b \leq 2, \tag{14}$$

however since there is no trade in assets, there are separate asset market clearing conditions for the two countries, as under autarky:

$$b_t^i = 0, \quad s_t^i = 1.$$

(f) As the problem states, without international borrowing the additional condition that an allocation must satisfy is, for $i = a, b$:

$$p_1 c_1^i + c_2^i = p_1 e_1^i + 1.$$

Note also that from (11) we know that the equalization of the goods price across countries implies that the ratio of c_2/c_1 is also equalized across countries. Using these conditions and the goods market clearing conditions (14) at equality we find the price of good 1:

$$p_1 = \frac{2\alpha}{1-\alpha}.$$

This in turn gives the consumption allocation:

$$\begin{aligned} c_1^i &= \alpha e_1^i + \frac{1}{2}(1-\alpha) \\ c_2^i &= 2\alpha e_1^i + 1 - \alpha. \end{aligned}$$

Using this allocation in (7) and (13) gives the expression for the risk-free rates:

$$\frac{1}{1+r_g^i} = \beta E\left(\left(\frac{2\alpha(e_1^i)' + 1 - \alpha}{2\alpha e_1^i + 1 - \alpha}\right)^{-\gamma} \Big| e_1^i\right)$$

As before, we can use the structure of the problem to be a little more explicit in the expressions. For country a we get:

$$\begin{aligned}\frac{1}{1+r_g^a} &= \beta\left(p+(1-p)\left(\frac{2\alpha(1-e_h)+1-\alpha}{2\alpha e_h+1-\alpha}\right)^{-\gamma}\right) \quad \text{if } e_1^a = e_h \\ &= \beta\left(p\left(\frac{2\alpha e_h+1-\alpha}{2\alpha(1-e_h)+1-\alpha}\right)^{-\gamma}+(1-p)\right) \quad \text{if } e_1^a = 1-e_h\end{aligned}$$

Similar expressions hold for country b, and the interest rates will vary across countries and across states.

(g) Using the parameter values and our previous expressions, we get:

$$\begin{aligned}\frac{1}{1+r_A^a} &= \beta\left(p+3(1-p)\right) \quad \text{if } e_1^a = e_h \\ &= \beta\left(\frac{1}{3}p+(1-p)\right) \quad \text{if } e_1^a = 1-e_h \\ \frac{1}{1+r_g^a} &= \frac{1}{1+r_f} = \beta\left(p+\left(\frac{5}{3}\right)^3(1-p)\right) \quad \text{if } e_1^a = e_h \\ &= \beta\left(\left(\frac{3}{5}\right)^3 p+(1-p)\right) \quad \text{if } e_1^a = 1-e_h \\ \frac{1}{1+r_f^a} &= \beta.\end{aligned}$$

Since $\left(\frac{5}{3}\right)^3 > 3$ we find that interest rates are more volatile (lower in the high state, and higher in the low state) when goods trade is allowed than in autarky. Under these parameter values, the intertemporal elasticity of substitution $(1/\gamma)$ is rather low so that the smoothing of the consumption allocation that comes with goods trade leads to more volatile asset prices (through the volatility of marginal utility). The interesting thing is that this is not monotone, as clearly there are no fluctuations in interest rates when there is free trade in goods and assets.

2

(a) The Bellman equation is

$$V(k,G) = \max_c\left\{\frac{(cG^\eta)^{1-\gamma}}{1-\gamma}+\beta V\left((1-\delta)k+f(k)-c,G'\right)\right\}.$$

The first order and envelope coniditons are

$$\begin{aligned}(cG^\eta)^{-\gamma}G^\eta &= \beta V_k(k',G') &(15)\\ V_k(k,G) &= \beta V_k(k',G')\left[(1-\delta)+f'(k)\right] \\ &= \left[(1-\delta)+f'(k)\right](cG^\eta)^{-\gamma}G^\eta. &(16)\end{aligned}$$

Combining the two equations we get the Euler equation

$$(cG^{\eta})^{-\gamma} G^{\eta} = \beta [(1-\delta) + f'(k')] (c'G'^{\eta})^{-\gamma} G'^{\eta}.$$

So we have the following dynamic system:

$$\begin{aligned} c_{t+1} &= c_t \left(\frac{G_{t+1}}{G_t}\right)^{\eta\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} [(1-\delta) + f'(k_{t+1})]^{\frac{1}{\gamma}} && (17) \\ k_{t+1} &= (1-\delta) k_t + f(k_t) - c_t. && (18) \end{aligned}$$

(b) If G_t grows at a constant rate g, i.e., $G_{t+1} = (1+g) G_t$, then (17) becomes

$$c_{t+1} = c_t (1+g)^{\eta\frac{1-\gamma}{\gamma}} \beta^{\frac{1}{\gamma}} [(1-\delta) + f'(k_{t+1})]^{\frac{1}{\gamma}}. \quad (19)$$

At a steady state we have $k_{t+1} = k_t = k_{ss}$ and $c_{t+1} = c_t = c_{ss}$, so (18) and (19) yield

$$\begin{aligned} f'(k_{ss}) &= \frac{(1+g)^{-\eta(1-\gamma)}}{\beta} - (1-\delta) \\ c_{ss} &= f(k_{ss}) - \delta k_{ss}. \end{aligned}$$

Assuming that $f' > 0$, and $f'' < 0$, which are necessary for a neoclassical production function, the above steady state is unique, since f' is 1-to-1. In this case the steady state is

$$\begin{aligned} k_{ss} &= (f')^{-1} \left(\frac{(1+g)^{-\eta(1-\gamma)}}{\beta} - (1-\delta)\right) && (20) \\ c_{ss} &= f(k_{ss}) - \delta k_{ss}. && (21) \end{aligned}$$

(c) To see what happens in the short run we will need to write down the $\Delta c = 0$ and $\Delta k = 0$ curves. They are, respectively,

$$\begin{aligned} k &= (f')^{-1} \left(\frac{(1+g)^{-\eta(1-\gamma)}}{\beta} - (1-\delta)\right) && (22) \\ c &= f(k) - \delta k, && (23) \end{aligned}$$

where we have assumed that $f'(\cdot)$ is invertible. To see what happens in the long run we will just need to look at the steady state. We have two cases:

i) $\eta > 0$:
From (22) we see that a higher g implies the $\Delta c = 0$ line moves to the left, while from (23) we see that the $\Delta k = 0$ curve is unchanged. This is shown in Figure 1.

Also directly from the above graph or from (20) we see that a higher g implies a lower steady state stock of capital, since $(f')^{-1}(\cdot)$ is decreasing,

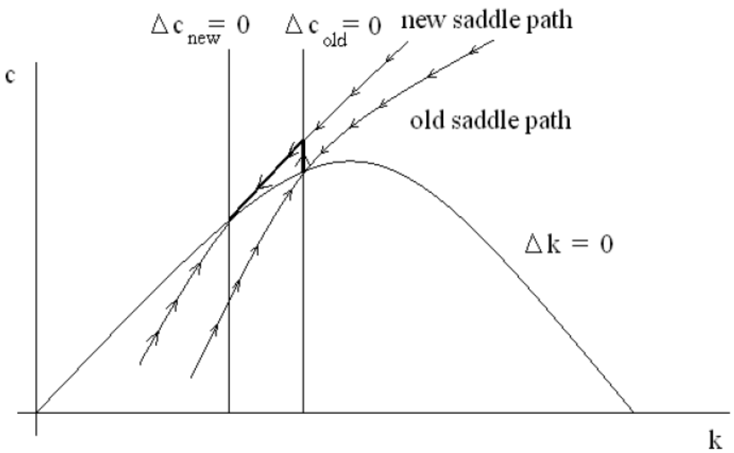

Figure 1: Effect of increase in g on the curves $\Delta c = 0$ and $\Delta k = 0$ and on the saddle path, and transition from the old steady state to the new one, for $\eta > 0$.

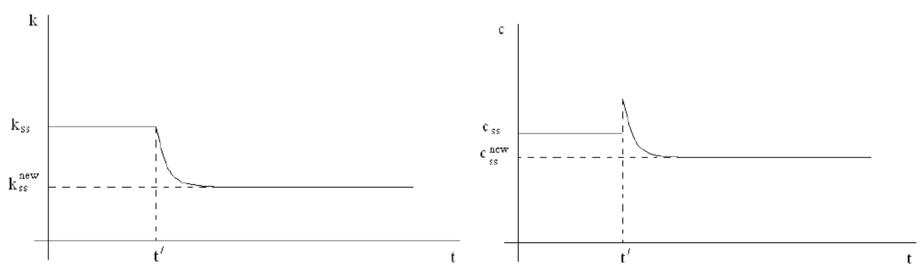

Figure 2: Left: Evolution of k after increase in g, for $\eta > 0$. Right: Evolution of c after increase in g, for $\eta > 0$.

and hence from (21) a lower steady state consumption. Combining all the above, we can figure out the transition dynamics qualitatively. Assuming that when the unexpected increase in g occurred, the economy was at a steady state, the transition dynamics are given by the Figure 2.

ii) $\eta < 0$:
From (22) we see that a higher g implies the $\Delta c = 0$ line moves to the right, while from (23) we see that the $\Delta k = 0$ curve is unchanged. This is shown in Figure 3.

Also directly from the above graph or from (20) we see that a higher g implies a higher steady state stock of capital, since $(f')^{-1}(\cdot)$ is decreasing, and hence from (21) a higher steady state consumption. Combining all the above, we can figure out the transition dynamics qualitatively. Assuming that when the unexpected increase in g occurred, the economy was at a

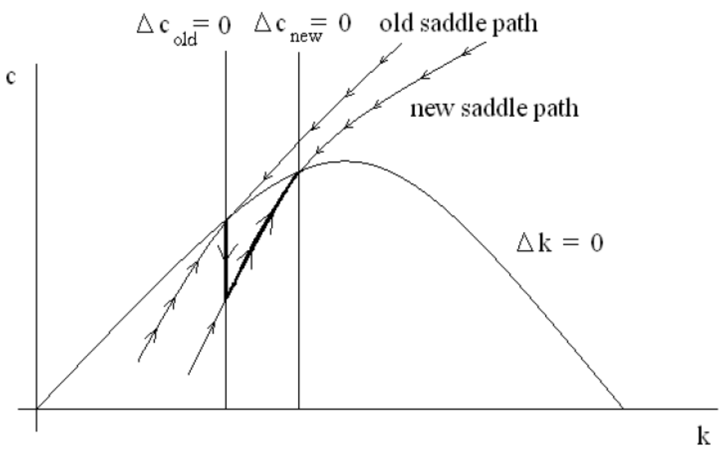

Figure 3: Effect of increase in g on the curves $\Delta c = 0$ and $\Delta k = 0$ and on the saddle path, and transition from the old steady state to the new one, for $\eta < 0$.

steady state, the transition dynamics are given by the Figure 4.

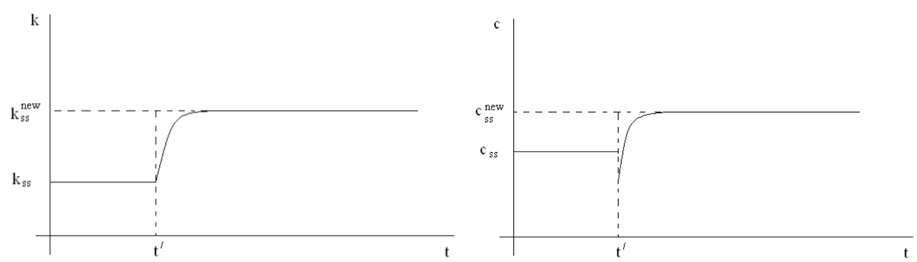

Figure 4: Left: Evolution of k after increase in g, for $\eta < 0$. Right: Evolution of c after increase in g, for $\eta < 0$.

3

(a) Although households take prices, taxes and transfers as given, it must be able to project next period's prices and taxes in order to solve its problem. So we need to introduce a *perceived law of motion*, which we do by assuming rational expections. To do this, define the aggregate state as the vector $X = [K\ z]$.

A recursive competitive equilibrium (RCE) with a government is a value function $V(k; X)$, a set of decision rules $\{c(k; X), k'(k; X), n(k; X)\}$, a set of prices $\{r(X), w(X)\}$, transfers $\{T(X)\}$, and a perceived law of motion for capital $K'(X)$, given the tax rate τ and transistion function $P(z', z)$ and $N = 1$, such that

(i) Given prices, taxes, transfers and the law of motion, the decision rules solve the household's problem:

$$V(k;X) = \max_{c,k'} u(c) + \beta \int_{z'} V(k';K'(X),z')dP(z,z')$$
$$c + k' = (1-\tau)(w(X)n + r(X)k) + (1-\delta)k + T(X)$$

(ii) For all $X = [K\ z]$, (K, N) solves the representative firm's problem given prices:

$$zF_K(K,N) = r(X)$$
$$zF_N(K,N) = w(X)$$

(iii) For all X, the government balances budget:

$$T(X) = \tau(w(X)N + r(X)K)$$

(iv) Markets clear, i.e. $k'(k;X) = K'(X)$ and $n(k;X) = N = 1$ for all $[k;X]$,

(v) $k = K$, i.e. aggregate state equals individual state. This is required due to the representative agent setting.

(vi) The law of motion is induced by

$$K'(X) = zF(K,N) + (1-\delta)K - c(K;X) = k'(K;X),$$

(b) Given today's aggregate state X and the law of motion $K'(X)$, the household rationally projects next period's aggregate state as $X' = [K'(X), z']$. Since there is no preferences for leisure, $n(k;X) = 1$ for all k, X. To write a FE in terms of $k'(k;X)$, it will be easier to let the agent's state variable be his wealth, which in turn is a function of $[k\ X]$:

$$a(k;X) \equiv (1-\tau)[w(X) + r(X)k] + (1-\delta)k + T(X),$$

so from the perspective of the agent who chooses k' but forecasts X', it faces the individual law of motion:

$$a(k';X') = (1-\tau)[w(X') + r(X')k'] + (1-\delta)k' + T(X'). \qquad (24)$$

Now let $V(a(k;X), z) \equiv V(k;X)$, the Bellman equation for the HH is

$$V(a(k;X),z) = \max_{k'} \left\{ u(a(k;X) - k') + \beta \int_{z'} V(a(k';X'),z')dP(z,z') \right\}$$

subject to the law of motion (24). The f.o.c. at the solution is

$$u'(a(k;X) - k'(k;X)) = \beta \int_{z'} [(1-\tau)r(G(X),z') + 1 - \delta] \cdot \\ V_a(a(k'(k;X);X'),z')dP(z,z')$$

and combining with the envelope condition

$$V_a(a(k;X);X) = u'(a(k;X) - k'(k;X))$$

we get the Euler equation

$$u'(a(k;X) - k'(k;X)) = \beta \int_{z'} [(1-\tau)r(X') + 1 - \delta] \cdot \\ u'\left[a(k'(k;X);X') - k'(k'(k;X);X')\right] dP(z,z').$$

Now being explicit about rational expectations and the individual's state variables, we rewrite

$$u'(a(k;K,z) - k'(k;K,z)) = \beta \int_{z'} [(1-\tau)r(K'(K,z),z') + 1 - \delta] \times \\ u'\left[a(k'(k;K,z);K'(K,z),z') - k'(k'(k;K,z);K'(K,z),z')\right] dP(z,z'). \quad (25)$$

Hence given a forecasting rule $K'(K,z)$, (25) defines a functional equation for $k'(k;K,z)$.

(c) From the firm's f.o.c.'s, in equilibrium given that $N = 1$,

$$r(K,z) = zF_K(K,1)$$
$$w(K,z) = zF_N(K,1)$$

Now we know in equilibrium that $k = K$ and $k'(K;K,z) = K'(K,z)$. So the representative agent's state in equilibrium is (assuming a HD1 technology)

$$a(k;K,z) = zF(K,1) + (1-\delta)K \equiv A(K,z)$$

and suppressing all the individual state variables in (25), we get

$$u'(A(K,z) - K'(K,z)) = \beta \int_{z'} [(1-\tau)z'F_K(K'(K,z),z') + 1 - \delta] \cdot \\ u'\left[A(K'(K,z),z') - K'(K'(K,z),z')\right] dP(z,z').$$

The whole idea is as follows. First, we can solve for $K'(K,z)$ from (a) to get the aggregate law of motion. Then, plug this in the HH problem as the forecast function for tomorrow's aggregate capital, and solve out for $k'(k,K,z)$ such that $k'(K,K,z) = K'(K,z)$.

(d) If $\delta = 1$, the RCE law of motion satisfies

$$u'(A(K,z) - K'(K,z)) = \tilde{\beta} \int_{z'} z' F_K(K'(K,z), z') \cdot \\ u'\left[A(K'(K,z), z') - K'(K'(K,z), z')\right] dP(z, z').$$

where $\tilde{\beta} = \beta(1-\tau)$. Hence the problem coincides with a planner's problem with discount factor $\tilde{\beta}$. The planner who internalizes government budget balance would not choose the distorted allocation. But since taxes are constant and $\delta = 1$, the allocation coincides exactly with a planner who faces a discount factor of $\tilde{\beta}$. So the distortionary tax has the effect of making the economy act as if were more impatient, and hence reduces capital accumulation (savings).

4

(a) By (t) denote function estimated with variables at time t.

From lecture notes, HH optimality conditions with capital and labor taxes are:

$$u_c(t) = \beta u_c(t+1)[1 + (1 - \tau^k_{t+1})(r_{t+1} - \delta)]$$

$$\frac{u_l(t)}{u_c(t)} = (1 - \tau^n_t) w_t$$

Lagrangian of the HH problem with consumption tax:

$$L = \sum \beta^t u(c_t, 1 - n_t) + \lambda \left[\sum q_t \left(w_t n_t + r_t k_t - (1 + \tau^c_t) c_t - k_{t+1} + (1 - \delta) k_t \right) \right]$$

First order conditions:

$$[c_t] : \beta^t u_c(t) = \lambda q_t (1 + \tau^c_t)$$

$$[n_t] : \beta^t u_l(t) = \lambda q_t w_t$$

$$[k_{t+1}] : q_t = q_{t+1}(1 + r_{t+1} - \delta)$$

Substitute out λ to get

$$u_c(t) = \beta u_c(t+1)(1 + r_{t+1} - \delta) \frac{1 + \tau^c_t}{1 + \tau^c_{t+1}} \qquad \text{(EE)}$$

$$\frac{u_l(t)}{u_c(t)} = \frac{w_t}{1 + \tau^c_t}$$

To induce the same allocation as with consumption tax, labor and capital taxes must satisfy

$$1 + (1 - \tau^k_{t+1})(r_{t+1} - \delta) = (1 + r_{t+1} - \delta) \frac{1 + \tau^c_t}{1 + \tau^c_{t+1}}$$

$$1 - \tau^n_t = \frac{1}{1 + \tau^c_t}$$

For any sequence τ_t^c, it is possible to construct sequences τ_t^n and τ_t^k that will satisfy these conditions. In general, τ_t^k will be a function of r_t. In a special case of $\tau_t^c = const$, $\tau_t^k = 0$.

(b) With $\frac{1+\tau_{t+1}^c}{1+\tau_t^c} = 1 + g$ and $r_t = F_k(t)$, (EE) becomes

$$(1+g)u_c(t) = \beta u_c(t+1)(1 + F_k(t+1) - \delta)$$

Steady state lines of the phase diagram then are:

$$[c_t = c_{t+1}] : 1 + g = \beta(1 + F_k(t+1) - \delta)$$
$$[k_t = k_{t+1}] : c_t = F(t) - \delta k_t - G$$

Increase in g shifts the vertical $c_t = c_{t+1}$ line to the left, and increase in G shifts the $k_t = k_{t+1}$ curve down. New steady state capital and consumption levels are lower.

In transition, the system jumps to the new saddle path and converges to the new steady state. Capital is gradually declining, and consumption starts declining after initial jump. This initial jump can be positive or negative, depending on whether the new saddle path is above or below the old steady state.

(c) We will use primal approach. From HH FOCs and $q_0 = 1$ get equations to substitute out prices and taxes:

$$q_t(1 + \tau_t^c) = \beta^t \frac{u_c(t)}{u_c(0)}(1 + \tau_0) \tag{26}$$

$$q_t w_t = \beta^t \frac{u_l(t)}{u_c(0)}(1 + \tau_0) \tag{27}$$

Using no arbitrage condition $1 + r_t - \delta = q_t/q_{t+1}$, rewrite budget constraint as

$$\sum q_t((1 + \tau_t^c)c_t - w_t n_t) = q_0 k_0(1 + r_0 - \delta)$$

With $r_t = F_k(t)$, $q_0 = 1$ and equations (26) and (27) budget constraint becomes

$$\sum \beta^t(u_c(t)c_t - u_l(t)n_t) = k_0(1 + F_k(0) - \delta)$$

Now the Ramsey problem can be formulated as a choice of only allocations, and the rest of its solution is identical with lecture notes. We cannot get much further than FOC characterization without specific functional forms.

Denote by $\bar{x}$ a steady state level of variable x_t. From the solution of the Ramsey problem one can obtain $\bar{k}, \bar{n}$ and $\bar{c}$. Then $\bar{w} = F_n(\bar{k}, \bar{n})$. Divide (26) by (27) to get

$$\frac{1 + \tau_t^c}{w_t} = \frac{u_c(t)}{u_l(t)}$$

In steady state consumption tax is constant:

$$1 + \tau_t^c = F_n(\bar{k}, \bar{n}) \frac{u_c(\bar{c}, 1 - \bar{n})}{u_l(\bar{c}, 1 - \bar{n})}$$