Econ 714: Problem Set 3 - Solution[1]

1 [2]

(a) Maximize expected utility subject to resource and incentive compatibility constraints:

$$\max_{c_1(\theta),c_2(\theta),x(\theta)} \mathbb{E}\left[u(c_1)+\beta(u(c_2)-v(l))\right]$$

$$\text{s.t. } x=\theta l$$

$$\sum_i p_i(c_1(\theta_i)+c_2(\theta_i)) = y_1+\sum_i p_i x(\theta_i)$$

$$\mathbb{E}\left[u(c_1(\theta_H))+\beta(u(c_2(\theta_H))-v(x(\theta_H)/\theta_H))\right] \geq \mathbb{E}\left[u(c_1(\theta_L))+\beta(u(c_2(\theta_L))-v(x(\theta_L)/\theta_H))\right]$$

$$\mathbb{E}\left[u(c_1(\theta_L))+\beta(u(c_2(\theta_L))-v(x(\theta_L)/\theta_L))\right] \geq \mathbb{E}\left[u(c_1(\theta_H))+\beta(u(c_2(\theta_H))-v(x(\theta_H)/\theta_L))\right]$$

where $p_i = 1/2$ for $i = L, H$.

(b) We need to show that $u'(c_1(\theta_i)) = \beta u'(c_2(\theta_i))$.

Suppose that it's not true and, without loss of generality, $u'(c_1(\theta_i)) > \beta u'(c_2(\theta_i))$. Then it is possible to find another allocation $c_1'(\theta_i) > c_1(\theta_i), c_2'(\theta_i) < c_2(\theta_i)$ that satisfies $u(c_1'(\theta_i)) + \beta u(c_2'(\theta_i)) = u(c_1(\theta_i)) + \beta u(c_2(\theta_i))$ and $c_1'(\theta_i) + c_2'(\theta_i) < c_1(\theta_i) + c_2(\theta_i)$. Since new allocation gives same utility, all incentive constraints are satisfied. But there are now extra resources available, that can be used to increase all types' utilities by the same amount, increasing welfare.

Hence, conjecture $u'(c_1(\theta_i)) > \beta u'(c_2(\theta_i))$ is wrong, and it must be that $u'(c_1(\theta_i)) = \beta u'(c_2(\theta_i))$.

(c) We need to show that $\frac{v'(x(\theta_H)/\theta_H)}{u'(c_2(\theta_H))} = \theta_H$.

Suppose it's not true and, without loss of generality, $\frac{v'(x(\theta_H)/\theta_H)}{u'(c_2(\theta_H))} > \theta_H$. Consider another allocation $x'(\theta_H) = x(\theta_H) - \Delta_x$, $c_2'(\theta_H) = c_2(\theta_H) - \Delta_c$. Choose Δ_x and Δ_c so that $u(c_2'(\theta_H)) - v(x'(\theta_H)/\theta_H) = u(c_2(\theta_H)) - v(x(\theta_H)/\theta_H)$. For that to hold, we need

$$u'(c_2(\theta_H))\Delta_c = v'(x(\theta_H)/\theta_H)\Delta_x/\theta_H$$

or

$$\frac{\Delta_c}{\Delta_x} = \frac{v'(x(\theta_H)/\theta_H)}{\theta_H u'(c_2(\theta_H))} > 1$$

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[1] By Anton Babkin. March 8, 2016.

[2] For a more general case see "Wedges and Taxes" by Kocherlakota (2004, AER Papers and Proceedings), and "Optimal Indirect and Capital Taxation" by Golosov et al. (2003, Review of Economic Studies).

The last inequality follows from out conjecture. So $\Delta_c > \Delta_x$, meaning that resource constraint can be relaxed again, allocation is not optimal, hence conjecture is wrong.

Notice that in this proof reduction in $x(\theta_H)$ increases RHS of the incentive constraint for the θ_L type. But as was mentioned in class, this constraint is not binding, hence we can always choose a change sufficiently small not to violate it.

(d) Suppose that $u'(c_1(\alpha)) = \beta \mathbb{E}_{\theta|\alpha} u'(c_2(\theta, \alpha))$ for a given α, i.e. standard Euler equation holds and intertemporal allocation is not distorted.

To simplify notation, omit α and denote $c_2(\theta_i, \alpha)$ by c_2^i.

Consider an allocation $c_1' = c_1 + \Delta/u'(c_1)$, $c_2'^i = c_2^i - \Delta/u'(c_2^i)$. For an infinitesimal Δ, this allocation also satisfies the above Euler equation, i.e. it is optimal. Also note that $u(c_1') + \beta \mathbb{E} u(c_2'^i) = u(c_1) + \beta \mathbb{E} u(c_2^i))$, so the new allocation satisfies all incentive contraints and does not change value of the welfare function.

By Jensen's inequality

$$\frac{1}{u'(c_1)} = \frac{1}{\mathbb{E} u'(c_2^i)} < \mathbb{E} \frac{1}{u'(c_2^i)} = \sum_i p_i \frac{1}{u'(c_2^i)}$$

Then

$$c_1' - c_1 = \frac{\Delta}{u'(c_1)} < \sum_i p_i \frac{\Delta}{u'(c_2^i)} = \sum_i p_i (c_2^i - c_2'^i)$$

Or

$$c_1' + \sum_i p_i c_2'^i < c_1 + \sum_i p_i c_2^i$$

Again, new allocation consumes less resources, meaning that welfare can be increased, and original allocation can not be optimal. Hence our conjecture is wrong and $u'(c_1(\alpha)) \neq \beta \mathbb{E}_{\theta|\alpha} u'(c_2(\theta, \alpha))$, i.e. intertemporal allocation must be distorted.

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Note: This solution uses the following corrections/clarifications:

- $E[\epsilon_{t+1}|x_t] = 0$,
- $g_{t+1} \equiv M_{t+1}/M_t$,
- $\frac{1}{g_t} \in [\frac{\epsilon}{1-\rho}, \frac{\bar{\epsilon}}{1-\rho}]$
- $R(x_{t-1})$ on the RHS of the HH budget constraint.

(a) $R(x_1) > 0$: nominal interest rate on bonds is positive. But money yields zero nominal return, so money is dominated by bonds as an asset. Hence, it is never optimal to hold excess money and CIA constraint must bind.

Household problem:

$$V(w_t) = \max_{c_t, s_t, B_t, M_t^d} u(c_t) + \beta \mathbb{E}[V(w_{t+1})|x_t]$$

$$\text{s.t. } w_t = (d_{t-1}\frac{P_{t-1}}{P_t} + p(x_t))s_{t-1} + \frac{B_{t-1}R(x_{t-1})}{P_t} + \frac{M_{t-1}^d - P_{t-1}c_{t-1}}{P_t} \quad \forall t$$

$$c_t - \tau_t + p(x_t)s_t + \frac{B_t}{P_t} \leq w_t \quad \forall t$$

$$P_t c_t \leq M_t^d \quad \forall t$$

$$c_t \geq 0 \quad \forall t$$

Solution to this functional equation exists and is unique if $u(c_t)$ is bounded and continuous, domain of the vector of choice variables is convex, and feasibility correspondence $\Gamma(w_t)$ implied by constraints is nonempty, compact-valued and continuous.

Value function $V(w_t)$ is strictly increasing if $u(c_t)$ is strictly increasing in w_t, and $\Gamma(w_t)$ is monotone in w_t.

$V(w_t)$ is differentiable if $u(c_t)$ is strictly concave in $w_t, c_t, s_t, B_t, M_t^d$ and differentiable in w_t in the interior of the feasibility set, and $\Gamma(w_t)$ is convex.

(b) Recursive competitive equilibrium:

- Prices $P_t, p(x_t), R(x_t)$,
- Value function $V(w_t)$ and policy functions $c_t(w_t), s_t(w_t), B_t(w_t), M_t^d(w_t)$ that solve household problem for these prices and tax τ_t,
- Markets clear: $c_t = d_t$, $B_t = 0$, $M_t^d = M_t$, $s_t = 1$,
- Government budget holds: $\tau_t = \frac{M_{t+1} - M_t}{P_t}$.

(c) $c_t \geq 0$ never binds if we assume $u'(0) = \infty$, and budget constraint always binds if $u'(c_t) > 0$.

CIA constraint $P_t c_t \leq M_t^d$ always binds if we assume $R(x_t) > 1$, so we can substitute out M_t^d.

Rewrite household problem as:

$$V(w_t) = \max_{c_t, s_t, B_t} u(c_t) + \beta \mathbb{E}[V(w_{t+1})|x_t]$$

$$\text{s.t. } w_t = (d_{t-1}\frac{P_{t-1}}{P_t} + p(x_t))s_{t-1} + \frac{B_{t-1}R(x_{t-1})}{P_t} + \frac{M_{t-1}^d - P_{t-1}c_{t-1}}{P_t}$$

$$c_t = w_t + \tau_t - p(x_t)s_t - \frac{B_t}{P_t}$$

Plug in c_t and w_{t+1}, and find optimality conditions:

$$FOC[s_t] : u'(c_t)p(x_t) = \beta \mathbb{E} V'(w_{t+1})(d_t \frac{P_t}{P_{t+1}} + p(x_{t+1}))$$
$$FOC[B_t] : \frac{u'(c_t)}{P_t} = \beta \mathbb{E} V'(w_{t+1}) \frac{R(x_t)}{P_{t+1}}$$
$$ENV[w_t] : V'(w_t) = u'(c_t)$$

These imply two Euler equations that can be used to price bonds and Lucas tree:

$$u'(c_t)p(x_t) = \beta \mathbb{E} u'(c_{t+1})(d_t \frac{P_t}{P_{t+1}} + p(x_{t+1})) \qquad (EE_s)$$
$$\frac{u'(c_t)}{P_t} = \beta \mathbb{E} u'(c_{t+1}) \frac{R(x_t)}{P_{t+1}} \qquad (EE_B)$$

(d) We will guess and verify that $R(x_t) > 1$.

As was discussed above, under this guess CIA constraint binds. Use $u'(c_t) = 1/c_t$, $c_t = d_t$ and $P_t c_t = M_t$ in equation (EE_B):

$$\begin{aligned} \frac{1}{c_t P_t} &= \beta \mathbb{E} \frac{R(x_t)}{c_{t+1} P_{t+1}} \\ \frac{1}{R(x_t)} &= \beta \mathbb{E} \frac{c_t P_t}{c_{t+1} P_{t+1}} \\ &= \beta \mathbb{E} \frac{M_t}{M_{t+1}} \\ &= \beta \mathbb{E}[\rho \frac{1}{g_t} + \epsilon_{t+1}] \\ &= \beta \frac{\rho}{g_t} \\ R(x_t) &= g_t \frac{1}{\beta \rho} \end{aligned}$$

Under assumptions on the stochastic process for g_t, $\frac{1}{g_t} \leq \frac{\bar{\epsilon}}{1-\rho} < \frac{1}{\beta\rho}$, or $g_t > \beta\rho$. It follows that $R(x_t) > 1$, guess verified.

(e) Recursively substitute equation (EE_s):

$$
\begin{aligned}
\frac{p(x_t)}{c_t} &= \beta\mathbb{E}\frac{d_t P_t}{c_{t+1}P_{t+1}} + \beta\mathbb{E}\frac{p(x_{t+1})}{c_{t+1}} \\
p(x_t) &= \beta\mathbb{E}d_t\frac{c_t P_t}{c_{t+1}P_{t+1}} + \beta\mathbb{E}\frac{c_t}{c_{t+1}}p(x_{t+1}) \\
&= \beta\mathbb{E}d_t\frac{M_t}{M_{t+1}} + \beta^2\mathbb{E}d_{t+1}\frac{M_{t+1}}{M_{t+2}}\frac{c_t}{c_{t+1}} + \beta^2\mathbb{E}\frac{c_t}{c_{t+2}}p_{t+2} \\
&= ... \\
&= d_t\sum_{i=1}^{\infty}\beta^i\mathbb{E}\frac{M_{t+i-1}}{M_{t+i}} \\
&= d_t\sum_{i=1}^{\infty}\beta^i\rho^i\frac{1}{g_t} \\
&= d_t\frac{\beta\rho}{1-\beta\rho}\frac{M_{t-1}}{M_t}
\end{aligned}
$$

(f) From CIA constraints:

$$\frac{P_{t+1}}{P_t} = \frac{M_{t+1}}{M_t}\frac{d_{t+1}}{d_t}$$

And

$$p(x_t) = d_t\frac{\beta\rho}{1-\beta\rho}\frac{M_{t-1}}{M_t}$$

Clearly, for a given ρ increase in money growth rate $\frac{M_{t+1}}{M_t}$ increases inflation and decreases price of Lucas trees.

3 [3]

a. HJB equations

The value functions of agents who own a house, V_1, and those who don't, V_0, are given by

$$
\begin{aligned}
rV_1 &= u + \alpha(1-H)\pi_0\pi_1[p + V_0 - V_1] && (1) \\
rV_0 &= \alpha H\pi_0\pi_1[-p + V_1 - V_0]. && (2)
\end{aligned}
$$

Given these two value functions, we can subtract the second from the first in order to find the welfare gain, $\Delta \equiv V_1 - V_0$:

$$\Delta = \frac{u + \alpha\pi_0\pi_1 p}{r + \alpha\pi_0\pi_1}. \tag{3}$$

[3] June 2013 Macro Prelim Exam. Solution by Kyle Dempsey

b. Optimal buying and selling strategies

In order to characterize each type of agents buying and selling strategies, we just compare the value of the action to its alternative, 0. For sellers, we have

$$\pi_1 = \begin{cases} 1 & \text{if } p - \Delta \geq 0 \\ [0,1] & \text{if } p - \Delta = 0 \\ 0 & \text{if } p - \Delta \leq 0 \end{cases} \tag{4}$$

Given equation (3), we can evaluate the conditioning expression:

$$p - \Delta = \frac{(r + \alpha\pi_0\pi_1)p - u - \alpha\pi_o\pi_1 p}{r + \alpha\pi_0\pi_1} = \frac{rp - u}{r + \alpha\pi_0\pi_1} = 0 \iff p = \frac{u}{r} \tag{5}$$

The two equations above pin down the optimal selling strategy for agents with houses.

Turning to the potential buyers of houses, we can proceed in a similar fashion:

$$\pi_0 = \begin{cases} 1 & \text{if } \Delta - p \geq 0 \\ [0,1] & \text{if } \Delta - p = 0 \\ 0 & \text{if } \Delta - p \leq 0 \end{cases} \tag{6}$$

This equation, together with (5), determines the optimal strategy for agents without houses. Note that under the assumption that agents trade when they are indifferent, we can set $\pi_0 = \pi_1 = 1$ when $\Delta = p$.

c. Equilibrium price and effects of housing supply

In order to determine the equilibrium house price p, we can consider equilibria in which houses trade, i.e. ones in which $\pi \equiv \pi_0\pi_1 > 0$. Given the equations for π_0 and π_1 in part b above, this is clearly only possible when $\Delta = p$, and so the equilibrium price is given by:

$$p = \frac{u}{r} \tag{7}$$

Observe that the price of houses p does not depend on housing supply H! This result makes sense because the price is determined in bilateral meetings, where regardless of the supply of houses in the overall economy one agent has a house and another does not. That is, **conditional on meeting**, the aggregate housing stock does not matter.