Macro prelim solutions - August 2013[1]

Disclaimer: These are unofficial solutions, they might have errors and be incomplete. Your comments and corrections are welcome.

Question 1.A.

I am getting the opposite result in this problem: risk-free interest rate rises over time.

(a) Recursive formulation of agent's problem:

$$V(a_t, A_t, b_t) = \max \log(c_t) + \beta \mathbb{E}_t V(a_{t+1}, A_{t+1}, b_{t+1})$$
$$\text{s.t. } c_t + p_t a_{t+1} + P_t A_{t+1} + q_t b_{t+1} = (p_t + d_t)a_t + (P_t + D_t)A_t + b_t$$

where asset a_t has price p_t and pays stochastic dividend d_t, asset A_t has price P_t and pays constant dividend $D_t = d_0$, and b_t is a risk-free bond with price q_t.

Recursive competitive equilibrium is sequence of quantities c_t, a_t, A_t, b_t and prices p_t, P_t, q_t such that:

- c_t, a_t, A_t, b_t solve agent's problem taking prices as given,
- markets clear: $a_t = 1$, $A_t = 1$, $b_t = 0$ and $c_t = d_t + D_t$.

It is not necessary to introduce risk-free bond into the model, risk-free rate can be computed without it using pricing kernel.

(b) Standard asset pricing Euler equations:

$$p_t = \beta \mathbb{E}_t \frac{c_t}{c_{t+1}}(p_{t+1} + d_{t+1})$$
$$P_t = \beta \mathbb{E}_t \frac{c_t}{c_{t+1}}(P_{t+1} + D_{t+1})$$
$$q_t = \beta \mathbb{E}_t \frac{c_t}{c_{t+1}}$$

With market clearing conditions, risk-free bond price is

$$q_t = \beta \mathbb{E}_t \frac{d_t + D_t}{d_{t+1} + D_{t+1}} = \beta \mathbb{E}_t \frac{d_t + d_0}{d_{t+1} + d_0}$$

Using law of motion for d_t, $E_t(f(d_{t+1})) = \pi f(d_t(1+\delta)) + (1-\pi)f(d_t)$. Then

$$\begin{aligned} q_t &= \beta[\pi \frac{d_t + d_0}{d_t(1+\delta) + d_0} + (1-\pi)\frac{d_t + d_0}{d_t + d_0}] \\ &= \beta[\pi \frac{d_t + d_0}{d_t(1+\delta) + d_0} + (1-\pi)] \\ &= \beta[\pi \frac{d_t(1+\delta) + d_0 - d_t\delta}{d_t(1+\delta) + d_0} + (1-\pi)] \\ &= \beta[1 - \pi \frac{d_t\delta}{d_t(1+\delta) + d_0}] \end{aligned}$$

[1]By Anton Babkin. This version: May 26, 2016.

In expectation as of $t = 0$, future change in price is

$$
\begin{aligned}
\mathbb{E}_0[q_{t+1} - q_t] &= \mathbb{E}_0\left[\mathbb{E}_t[q_{t+1} - q_t]\right] \\
&= \mathbb{E}_0\left[\mathbb{E}_t\left(\beta[1 - \pi\frac{d_{t+1}\delta}{d_{t+1}(1+\delta)+d_0}]\right) - \beta[1 - \pi\frac{d_t\delta}{d_t(1+\delta)+d_0}]\right] \\
&= \mathbb{E}_0\left[\beta\pi\left(\frac{d_t\delta}{d_t(1+\delta)+d_0} - \mathbb{E}_t[\frac{d_{t+1}\delta}{d_{t+1}(1+\delta)+d_0}]\right)\right] \\
&= \mathbb{E}_0\left[\beta\pi\left(\frac{d_t\delta}{d_t(1+\delta)+d_0} - \pi\frac{d_t\delta(1+\delta)}{d_t(1+\delta)^2+d_0} - (1-\pi)\frac{d_t\delta}{d_t(1+\delta)+d_0}\right)\right] \\
&= \mathbb{E}_0\left[\beta\pi^2\left(\frac{d_t\delta}{d_t(1+\delta)+d_0} - \frac{d_t\delta(1+\delta)}{d_t(1+\delta)^2+d_0}\right)\right] \\
&= \mathbb{E}_0\left[\beta\pi^2\left(\frac{d_t\delta(1+\delta)}{d_t(1+\delta)^2+d_0(1+\delta)} - \frac{d_t\delta(1+\delta)}{d_t(1+\delta)^2+d_0}\right)\right]
\end{aligned}
$$

$$
\begin{gathered}
d_t(1+\delta)^2 + d_0(1+\delta) > d_t(1+\delta)^2 + d_0 \\
\frac{d_t\delta(1+\delta)}{d_t(1+\delta)^2+d_0(1+\delta)} < \frac{d_t\delta(1+\delta)}{d_t(1+\delta)^2+d_0} \\
\mathbb{E}_0[q_{t+1} - q_t] < 0
\end{gathered}
$$

$\mathbb{E}_0 q_{t+1} < \mathbb{E}_0 q_t$, risk-free bond price q_t falls over time in expectation, so risk-free interest rate increases over time. Intuitively, asset a is becoming more and more valuable over time as it pays increasing dividend, so return on an alternative asset such as risk-free bond b must be also increasing so that it's market clears in equilibrium.

(c) Using the law of iterated expectations,

$$
\begin{aligned}
\mathbb{E}_0 d_t &= \mathbb{E}_0\mathbb{E}_{t-1} d_t \\
&= \mathbb{E}_0 d_{t-1}(\pi(1+\delta) + (1-\pi)) \\
&= \mathbb{E}_0\mathbb{E}_{t-2} d_{t-1}(1+\pi\delta) \\
&= \mathbb{E}_0 d_{t-2}(1+\pi\delta)^2 \\
&\dots \\
&= d_0(1+\pi\delta)^t
\end{aligned}
$$

So $\lim_{t\to\infty} \mathbb{E}_0 d_t = \infty$, in expectation dividend grows without bound.

Then the limit of the bond price is:

$$
\begin{aligned}
\lim_{t\to\infty} \mathbb{E}_0 q_t &= \lim_{t\to\infty} \mathbb{E}_0 \beta[1 - \pi\frac{d_t\delta}{d_t(1+\delta)+d_0}] \\
&= \lim_{t\to\infty} \mathbb{E}_0 \beta[1 - \pi\frac{\delta}{(1+\delta)+d_0/d_t}] \\
&= \beta[1 - \pi\frac{\delta}{1+\delta}] < \beta
\end{aligned}
$$

So in the limit the risk-free rate is $> 1/\beta$.

(d) Iterating Euler equation forward and assuming no-bubble equilibrium, obtain

$$\begin{aligned}\frac{p_t}{d_t} &= \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \frac{d_t + d_0}{d_{t+j} + d_0} \frac{d_{t+j}}{d_t} \\ &= \frac{d_t + d_0}{d_t} \mathbb{E}_t \sum_{j=1}^{\infty} \beta^j \frac{d_{t+j}}{d_{t+j} + d_0}\end{aligned}$$

As t increases, $\frac{d_t+d_0}{d_t}$ decreases at a decelerating rate. So the term $\frac{d_t+d_0}{d_t}$ decreases faster than each of the $\frac{d_{t+j}}{d_{t+j}+d_0}$ terms is increasing. Then price to dividend ratio is declining. In the limit $d_{t+j} \approx d_{t+j} + d_0$, so

$$\lim_{t\to\infty} \mathbb{E}_0 \frac{p_t}{d_t} = \sum_{j=1}^{\infty} \beta^j = \frac{\beta}{1-\beta}$$