1. Consider the following variant of Diamond's (1965) two period overlapping generations model with zero population growth. An agent born in period t chooses how much labor to supply in period t (denoted $n_t \in [0,1]$) and how much to save in period t (denoted $s_t \geq 0$) in order to maximize preferences given by

$$\log(c_t^t) - \frac{\gamma}{2} n_t^2 + \beta \log(c_{t+1}^t)$$

where c_t^t and c_{t+1}^t is consumption of the generation t agent in youth and old age, respectively, and $\beta \in (0,1)$. The production technology is given by the Cobb-Douglas function:

$$F(K_t, L_t) = K_t^{\alpha} L_t^{1-\alpha}$$

where $\alpha \in (0,1)$. There is 100% depreciation of capital. The initial old hold the capital stock K_0 and maximize utility $\log(c_0^{-1})$. Assume $\alpha \leq \frac{\beta}{1+2\beta}$.

In the decentralized competitive version of this economy, firms choose how much capital to rent from old agents at rate R_t and labor to hire from young agents at price w_t in order to maximize profits. Profits are distributed to the old, which they take as given. Proportional wage taxes $\tau_t \in [0,1]$ are imposed on the young and revenues are lump sum rebated to the young so that transfers $T_t = \tau_t w_t n_t$.

The first question refers to the planner's problem.

(a) **[10 points]** Write down the planner's problem which weights all generations equally. Derive the planner's optimality conditions and solve for steady state optimal allocations of capital and labor. Note: do not concern yourself with whether the objective is finite since you can always use the overtaking criterion.

The next set of questions refer to the decentralized economy.

(b) **[10 points]** Write down the maximization problem of a representative household of generation t. Derive the intra- and inter- temporal conditions for optimality assuming an interior solution. Is the labor supply policy function upward sloping in wages?

(c) **[2.5 points]** Write down the maximization problem of a representative firm in period t. Derive the two first order conditions. What are profits in equilibrium?

(d) **[2.5 points]** Write down the market clearing conditions for this economy and define a competitive equilibrium.

(e) **[15 points]** What is the equilibrium law of motion for capital in the decentralized economy? Illustrate dynamics in (K_t, K_{t+1})? How many steady state values of the capital stock are there in this economy and what are they?

The final question compares allocations in the planner's problem and the decentralized economy.

(f) **[10 points]** Are steady state allocations in the competitive equilibrium you found in (e) identical to those in the planner's problem in (a)? If not, can the government use labor taxes to implement the planner's solution?

2. **Guess and Verify [35 points].** Consider a social planner who faces the problem

$$\max_{\{c_t,k_{t+1},l_t\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t\left[\ln c_t + b\ln(1-l_t)\right],\ 0<\beta<1$$

subject to

$$c_t + k_{t+1} = A_t k_t^{\alpha} l_t^{1-\alpha},\ 0<\alpha<1$$

where A is a stochastic productivity parameter evolving via a first order autoregressive process:

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1},\ 0<\rho<1$$

with $E_t\epsilon_{t+1} = 0$ for all t.

(a) Formulate the problem using Bellman's functional equation.
(b) Derive the Euler equations and the first order conditions.
(c) Derive a closed form solution for the value and policy function.
(d) Interpret the effects of an increase in ρ on the value and policy functions. Explain.

3. **Time to Build - Setting up Bellman's Equation [15 points].** Consider a social planner who faces the problem

$$\max_{\{c_t,k_{t+2},l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u(c_t),\ 0<\beta<1$$

subject to

$$c_t + s_t = A_t F(k_t, l_t)$$

$$k_{t+1} = (1-\delta)k_t + i_t$$

$$i_t = s_{t-1}.$$

The production technology has the time to build feature where investment made in period $t-1$, s_{t-1}, becomes productive capital stock in period $t+1$.

(a) Formulate Bellman's equation for the planner.
(b) Derive the first order conditions and the envelope conditions.
(c) Derive the Euler equation. Interpret it (i.e. explain to me very clearly what it says)

4. Consider a pure exchange endowment economy where a representative agent has preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, X_t).$$

where C_t is the agent's consumption and X_t is an external preference factor shock factor, such as an external habit, that the agent treats as exogenous and beyond his control. The economy has a single nonstorable consumption good (fruit), given off by productive units (trees) with net supply of 1. Owners of trees receive the stochastic fruit yield (i.e., dividend or endowment) S_t each period, which follows a Markov process with transition function $Q(S', S)$. So if the price of the tree is p_t and the agent's holdings of shares of trees are a_t, his flow budget constraint is:

$$C_t + p_t a_{t+1} = (p_t + S_t) a_t.$$

Suppose the preference shock factor X_t is correlated with S_t and has a law of motion:

$$X' = F(X, S, S').$$

(a) **[10 points]** As a first step toward solving for a recursive equilibrium, conjecture a Markov pricing function for the tree p_t as a function of the aggregate state or states. Write down the Bellman equation for the representative agent and find his optimality conditions. What assumptions allow you to do this?

(b) **[5 points]** Define a recursive competitive equilibrium, being specific about the objects which make it up.

(c) **[10 points]** Find an expression for the equilibrium price of the tree. In addition, suppose that a redundant claim to a risk-free bond is traded, and characterize the equilibrium risk free interest rate. Interpret your answers.

(d) **[5 points]** Now consider the special case with power utility preferences independent of the external shock:

$$u(C, X) = \frac{C^{1-\gamma}}{1-\gamma}.$$

In addition, suppose that the dividend process has log-normal growth, so if $s_t = \log S_t$:

$$s_{t+1} = s_t + g + v_{t+1}$$

where $g > 0$ is the constant mean growth rate and $v_{t+1} \sim N(0, \sigma^2)$ i.i.d. Find an expression for the equilibrium risk free rate.

(e) **[5 points]** Recall that if m_{t+1} is a stochastic discount factor, then any stochastic excess return R^e_{t+1} (over the risk free rate) satisfies the pricing equation:

$$0 = E_t(m_{t+1} R^e_{t+1}).$$

Using the relationships between the expectation of a product, covariance, and correlation, find an expression for the maximal Sharpe ratio $E_t(R^e_{t+1})/\sigma_t(R^e_{t+1})$ in terms of the statistical properties of the stochastic discount factor.

(f) **[5 points]** What is the maximal Sharpe ratio in the model of part (d) with power utility and log-normal growth? Recall that if a random variable Y is log normal, $\log Y \sim$

$N(\mu, \sigma^2)$ then $E(Y) = \exp(\mu + \sigma^2/2)$. How does this result bear on the equity premium puzzle?

(g) **[5 points]** Now suppose the dividend process continues to have log-normal growth but that preferences are now defined by:

$$u(C, X) = \frac{(CX)^{1-\gamma}}{1-\gamma},$$

where X is also log normal, with $x_t = \log X_t$ satisfying:

$$x_{t+1} = (1-\phi)\bar{x} + \phi x_t + \lambda(x_t)[s_{t+1} - s_t - g],$$

where ϕ and $\bar{x}$ are positive constants and $\lambda(x)$ is a positive function. Now find expressions for the equilibrium risk free rate and the maximal Sharpe ratio.

(h) **[5 points]** Could the preference specification in part (g) help to solve the equity premium and risk-free rate puzzles? Explain and interpret.

5. **Stochastic Fixed Adjustment Costs.** Consider the problem of a firm which invests subject to a stochastic fixed adjustment cost. At the start of the period, our firm is going to have capital k and productivity A. Its choice is over k', how much capital to have at the beginning of the following period. Before choosing k', a random variable ϕ is drawn. With probability $1-\lambda$ (independent across periods), $\phi = 1$, and any adjustments to the capital stock involve a fixed cost of fk. With probability λ, $\phi = 0$ and any fixed costs of investment can be avoided. The case we worked out in class is the special case in which $\lambda = 0$. Thus the period profits are

$$\frac{R-1}{\alpha R} A^{1-\alpha} (k')^{\alpha} - (k'-k) - 0 \cdot f \cdot k \cdot 1_{k \neq k'} \text{ with probability } \lambda$$
$$\frac{R-1}{\alpha R} A^{1-\alpha} (k')^{\alpha} - (k'-k) - 1 \cdot f \cdot k \cdot 1_{k \neq k'} \text{ with probability } 1-\lambda$$

Here R is the gross interest rate (also R^{-1} is the firm's discount factor); α is the capital share; and A is the firm's productivity (which evolves according to a first-order Markov process). As in the lecture notes, suppose that the productivity growth rate, $\frac{A'}{A}$, is equal to some random variable γ, drawn from an i.i.d. distribution; $\log \gamma$ has expected value equal to 0. Also, as in the lecture notes, capital k' is installed instantaneously.

(a) [**7 points**] Let $\hat{V}(A, k, \phi)$ denote the value of a firm with productivity A and capital stock k and fixed cost shock ϕ. Write the firm's Bellman Equation.

(b) [**7 points**] Let $z \equiv \frac{k}{A}, z' \equiv \frac{k'}{A}, \gamma \equiv \frac{A'}{A}$, and $V(A, z, \phi) \equiv \hat{V}(A, Az, \phi)$. Re-write the Bellman equation in terms of $V(A, z, \phi)$.

(c) [**7 points**] Let $v(z, \phi) \equiv \frac{V(A,z,\phi)}{A}$. Write out the investment problem of the firm in terms of this dual-state value function.

(d) [**7 points**] Compute the FOC for the case where $\phi = 0$ and use $\tilde{z}$ to denote the optimal z' that satisfies this first order condition.

(e) [**7 points**] Conjecture that there exist three numbers $\underline{z} < z^* < \bar{z}$ such that
if $\phi = 1$ and $z \in [\underline{z}, \bar{z}]$ then $z' = z$,
if $\phi = 1$ and $z \notin (\underline{z}, \bar{z})$ then $z' = z^*$.
How do z^* and $\tilde{z}$ in part (d) compare?

(f) [**7 points**] For this part of the problem only, suppose that $\lambda = 1$, which implies that ϕ always equals 0. Compute z' and $q \equiv \frac{\partial v(z,\phi)}{\partial z}$ explicitly.

(g) [**8 points**] In the case in which $\lambda = 0$ (studied in class), what does the observed distribution of investment rates $z' - z$ look like? What if $\lambda \in (0, 1)$: how is the distribution of investment rates different?