Macro Example Comp Question from Quarter 1, 2015:
With solution

Analyze the dynamic equilibrium of a two period lived overlapping generations production economy with a constant unit measure of identical agents in each generation. Each agent's preferences are given by the utility function

$$u(c_{t+1}^t, n_t^t) = \frac{1}{1-\gamma}(c_{t+1}^t)^{1-\gamma} - n_t^t$$

where c_{t+1}^t denotes period $t+1$ consumption of an agent born in period t (i.e. an old person), $n_t^t \in [0,1]$ denotes labor supplied in period t by an agent born in period t (i.e. a young person). and $\gamma \in (0,1)$. Technology is given by the production function $f(k_t, n_t) = k_t^{1/2} n_t^{1/2}$ where k_t is total capital available to each producer in period t. Capital fully depreciates after production, so $k_{t+1} = i_t$ where i_t is investment chosen by the young in period t from their labor earnings. Young agents born in period t supply labor n_t^t at real wage w_t in order to buy capital k_{t+1} which they rent to firms in their second period of life at real gross return $R_{t+1}$to obtain funds $R_{t+1}k_{t+1}$for purchasing consumption goods c_{t+1}^t.

1. Write down the optimization problem faced by a generation t agent. Solve for labor supply and investment decision rules. (5 points)

   Answer:

   Capital depreciates fully, so $k_{t+1} = i_t$, and we can write that households choose capital directly. Firm ownership does not matter as technology is CRS, and profits will be zero in equilibrium.

$$\max_{c_{t+1}^t, n_t^t, k_{t+1}} \frac{1}{1-\gamma}(c_{t+1}^t)^{1-\gamma} - n_t^t$$
$$\text{s.t. } k_{t+1} = w_t n_t^t$$
$$c_{t+1}^t = R_{t+1}k_{t+1}$$
$$c_{t+1}^t, k_{t+1} \geq 0$$
$$n_t^t \in [0,1]$$

   Assume that parameters are such that $n_t^t \in (0,1)$, i.e. inequality constraints are not binding. Utility function satisfies Inada conditions, so $c_{t+1}^t \geq 0$ is not binding either. Using $c_{t+1} = R_{t+1}k_{t+1} = R_{t+1}w_t n_t^t$, rewrite household problem as

$$\max_{n_t^t} \frac{1}{1-\gamma}(R_{t+1}w_t n_t^t)^{1-\gamma} - n_t^t$$

   Take FOC in n_t^t:

$$R_{t+1}^{1-\gamma} w_t^{1-\gamma} (n_t^t)^{-\gamma} = 1$$

   Solve for labor supply, and plug into budget constraint for investment:

$$n_t^t = R_{t+1}^{\frac{1-\gamma}{\gamma}} w_t^{\frac{1-\gamma}{\gamma}}$$
$$k_{t+1} = R_{t+1}^{\frac{1-\gamma}{\gamma}} w_t^{\frac{1}{\gamma}}$$

2. Write down the optimization problem faced by a representative firm which rents labor at price w_t and capital at gross rate R_t to maximize real profits. (2.5 points)

   Answer:

   Firm's maximizes profit taking prices R_t, w_t as given:

$$\max_{K_t,N_t} y_t - R_t K_t - w_t N_t$$
$$\text{s.t. } y_t = K_t^{1/2} N_t^{1/2}$$

   Labor and capital demand desicions satisfy FOCs:

$$R_t = \frac{1}{2} K_t^{-\frac{1}{2}} N_t^{\frac{1}{2}}$$
$$w_t = \frac{1}{2} K_t^{\frac{1}{2}} N_t^{-\frac{1}{2}}$$

3. Define a competitive equilibrium. (2.5 points)

   Answer:

   Competitive equilibrium is an allocation $i_t, k_t, n_t, y_t, c_{t+1}^t$ and prices R_t, w_t such that $\forall t$:

   - Allocation solves household and firm optimization problems.
   - Markets clear:
     - labor: $n_t = N_t$,
     - capital: $k_t = K_t$,
     - goods: $c_t^{t-1} + i_t = y_t$

4. Show that a competitive equilibrium satisfies the following pair of first order difference equations

$$\begin{aligned} k_{t+1} &= R_t k_t, \\ R_{t+1}^{1-\gamma} &= 4 k_t^{\gamma} R_t^{1+\gamma}. \end{aligned}$$

   (10 points)

   Answer:

   Goods market clearing:

$$k_{t+1} + c_t^{t-1} = k_t^{\frac{1}{2}} n_t^{\frac{1}{2}}$$

   Using HH budget constraints and firm FOCs:

$$k_{t+1} + R_t k_t = k_t^{\frac{1}{2}} 2 R_t k_t^{\frac{1}{2}}$$

   This gives the first difference equation:

$$k_{t+1} = R_t k_t$$

Use firm FOCs to eliminate n and w from HH FOCs:

$$R_{t+1}^{1-\gamma}(\frac{1}{4R_t})^{1-\gamma}(4R_t^2k_t)^{-\gamma}=1$$

Simplify to get the second difference equation:

$$R_{t+1}^{1-\gamma}=4k_t^{\gamma}R_t^{1+\gamma}$$

5. Describe stationary competitive equilibria in (k_t, R_t) space and investigate their stability. (12.5 points)

   Answer:

   Dynamics of the model is characterised by a two-dimensional system of first order difference equations:

$$\begin{aligned} k_{t+1} &= R_tk_t \\ R_{t+1} &= 4^{\frac{1}{1-\gamma}}k_t^{\frac{\gamma}{1-\gamma}}R_t^{\frac{1+\gamma}{1-\gamma}} \end{aligned}$$

   or

$$\begin{pmatrix} k_{t+1} \\ R_{t+1} \end{pmatrix} = g(\begin{pmatrix} k_t \\ R_t \end{pmatrix})$$

   In steady state R and k solve

$$k=Rk$$
$$R^{1-\gamma}=4k^{\gamma}R^{1+\gamma}$$

   There are two solutions: $(k,R)=(0,0)$ and $(k,R)=(4^{-\frac{1}{\gamma}},1)$.

   To characterize stability of steady states we need matrix of first derivatives

$$Dg(k_t,R_t)=\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}=\begin{bmatrix} R_t & k_t \\ 4^{\frac{1}{1-\gamma}}k_t^{\frac{\gamma}{1-\gamma}}R_t^{\frac{1+\gamma}{1-\gamma}}\frac{\gamma}{1-\gamma}k_t^{-1} & 4^{\frac{1}{1-\gamma}}k_t^{\frac{\gamma}{1-\gamma}}R_t^{\frac{1+\gamma}{1-\gamma}}\frac{1+\gamma}{1-\gamma}R_t^{-1} \end{bmatrix}$$

   This is not well defined in $(0,0)$, so we can't characterize stability of the trivial steady state. We will focus on the second steady state, $(k,R)=(4^{-\frac{1}{\gamma}},1)$.

$$Dg(k,R)=\begin{bmatrix} 1 & 4^{-\frac{1}{\gamma}} \\ \frac{\gamma}{1-\gamma}4^{\frac{1}{\gamma}} & \frac{1+\gamma}{1-\gamma} \end{bmatrix}$$

   Eigenvalues λ of this matrix solve characteristic equation $p(\lambda)\equiv\lambda^2-T\lambda+D=0$, where

$$\begin{aligned} T &= g_{11}+g_{22}=\frac{2}{1-\gamma} \\ D &= g_{11}g_{22}-g_{21}g_{12}=\frac{1}{1-\gamma} \end{aligned}$$

Discriminant $\mathcal{D} = T^2 - 4D = \frac{4\gamma}{(1-\gamma)^2} > 0$, so there are 2 different real eigenvalues.

$$\lambda = \frac{\frac{2}{1-\gamma} \pm \sqrt{\frac{4\gamma}{(1-\gamma)^2}}}{2} = \frac{1 \pm \sqrt{\gamma}}{1-\gamma} = \frac{1}{1 \pm \sqrt{\gamma}}$$

So the two roots of the equation are such that $0 < \lambda_1 = \frac{1}{1+\sqrt{\gamma}} < 1$ and $\lambda_2 = \frac{1}{1-\sqrt{\gamma}} > 1$. Steady state is stable, and the system converges along a saddle path.

6. Suppose that the economy starts in a non-trivial steady state. Using a phase diagram, describe the dynamics of (k_t, R_t) after an unexpected negative shock to the capital stock (i.e. part of the capital accumulated by the young cohort is destroyed when they become old). Further, what happens to wages, labor supply, and consumption over time in response to the shock? (17.5 points)

   Answer:

$$\begin{aligned} k_{t+1} &= R_t k_t \\ R_{t+1} &= 4^{\frac{1}{1-\gamma}} k_t^{\frac{\gamma}{1-\gamma}} R_t^{\frac{1+\gamma}{1-\gamma}} \end{aligned}$$

$$k = const: \quad R_t = 1$$

k_t increases if $R_t > 1$ and decreases if $R_t < 1$.

$$R = const: \quad R = 4^{-\frac{1}{2\gamma}} k^{-\frac{1}{2}}$$

Rewrite the second difference equation as

$$\left(\frac{R_{t+1}}{R_t}\right)^{1+\gamma} = 4k_t^{\gamma} R_{t+1}^{2\gamma}$$

Above the $R = const$ line k is large and R is large, so $\frac{R_{t+1}}{R_t} > 0$ and R_t increases, below the line - decreases.

If capital drops below steady state level, R will immediatelly jump up to the saddle path. Over time, k will grow and R will decline back to the steady state.

Other variables of the model can be expressed as functions of the state vector (k_t, R_t):

$$\begin{aligned} n_t &= 4R_t^2 k_t \\ w_t &= \frac{1}{4R_t} \\ y_t &= k_t^{1/2} n_t^{1/2} = 2R_t k_t \\ c_t^{t-1} &= R_t k_t \end{aligned}$$

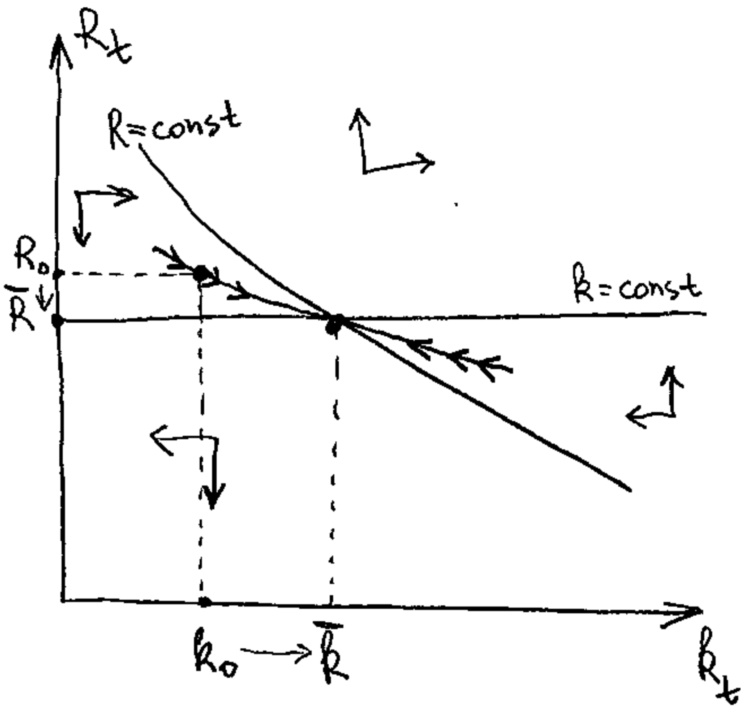

Figure 1: Phase diagram.

The first two equations follow from firm's FOCs, and the last one from household budget constraint. Clearly, on impact wage w_t will fall. Change in other variables depends on relative changes in k and R. Effects can be either computed using numerical methods, or evaluated with log-linear approximation around steady state.

First-order Taylor approximation of the system of difference equations is

$$\begin{pmatrix} k_{t+1} - k \\ R_{t+1} - R \end{pmatrix} = Dg(k, R) \begin{pmatrix} k_t - k \\ R_t - R \end{pmatrix}$$

Saddle path can be found as an eigenvector $\mathbf{v}$ that corresponds to eigenvalue $0 < \lambda_1 < 1$. $\mathbf{v}$ solves $(Dg(k, R) - I\lambda_1)\mathbf{v} = 0$. Normalize first element of $\mathbf{v}$ to $v_1 = 1$, so that $\mathbf{v} = (1, v_2)$.

$$\begin{bmatrix} 1 - \frac{1}{1+\sqrt{\gamma}} & 4^{-\frac{1}{\gamma}} \\ \frac{\gamma}{1-\gamma}4^{\frac{1}{\gamma}} & \frac{1+\gamma}{1-\gamma} - \frac{1}{1+\sqrt{\gamma}} \end{bmatrix} \begin{pmatrix} 1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} + 4^{-\frac{1}{\gamma}} v_2 = 0 \\ \frac{\gamma}{1-\gamma}4^{\frac{1}{\gamma}} + \frac{\sqrt{\gamma}}{1-\sqrt{\gamma}} v_2 = 0 \end{cases}$$

Two equations are identical, solving either one yields $v_2 = -\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}}4^{\frac{1}{\gamma}}$.

$k_0 - k$ is the initial shock to capital. Then R_0 must be on the saddle path. With linear approximation, slope of the saddle path is equal to the slope of the eigenvector, so

$$\frac{R_0 - R}{k_0 - k} = \frac{v_2}{v_1} = -\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} 4^{\frac{1}{\gamma}}$$

Let hats denote deviations from steady state, e.g. $\hat{R}_t = \frac{R_t - R}{R}$. Then

$$\hat{R}_0 = -\frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} \hat{k}_0$$

Log-linearize the four equations for other variables of the model:

$$\begin{aligned}
\hat{n}_t &= 2\hat{R}_t + \hat{k}_t \\
\hat{w}_t &= -\hat{R}_t \\
\hat{y}_t &= \hat{R}_t + \hat{k}_t \\
\hat{c}_t^{t-1} &= \hat{R}_t + \hat{k}_t
\end{aligned}$$

Plug in expression for $\hat{R}_0$, and with negative shock $\hat{k}_0 < 0$:

$$\begin{aligned}
\hat{n}_0 &= \frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}} \hat{k}_0 < 0 \\
\hat{w}_0 &= \frac{\sqrt{\gamma}}{1+\sqrt{\gamma}} \hat{k}_0 < 0 \\
\hat{y}_0 &= \frac{1}{1+\sqrt{\gamma}} \hat{k}_0 < 0 \\
\hat{c}_0^{-1} &= \frac{1}{1+\sqrt{\gamma}} \hat{k}_0 < 0
\end{aligned}$$

Labor, wages, output and consumption all decrease below their steady state levels in response to a negative shock to capital, and then gradually converge back.